Midterm 2 practice 2

UCLA: Math 3B, Fall 2017

Instructor: Noah White
Date: 27 February 2017

- This exam has 3 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section (please circle):

Day/TA	Kevin	Bohyun	Ryan
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	13	
2	14	
3	13	
Total:	40	

1. A large hole is filled with water. The hole has the shape of an upside down cone, with a depth of 5 meters at its deepest and a diameter of 20 meters at ground level. In this question we will calculate how much work is needed to pump all of the water out of the hole. You may assume that water has a density of 1000 kilograms per meter cubed and that the acceleration due to gravity is 10 m/s^2.

   (a) (1 point) Denote by x the depth below ground in meters, so that $x = 0$ is ground level and $x = 5$ is the bottom of the conical hole. We divide the hole into n horizontal slices. Let Δx be the thickness of each slice. What is Δx in terms of n?

   Solution:
   $$\Delta t = \frac{5}{n}$$

   (b) (1 point) Let x_k be the depth of the bottom of the k^{th} slice. What is x_k in terms of n and k?

   Solution:
   $$x_k = k\frac{5}{n} = k\Delta x$$

   (c) (2 points) The base of the k^{th} slice is a circle. What is its radius (leave your answer in terms of x_k)?

   Solution: Using the method of similar trianlges we find that the radius is $2(5 - x_k)$.

(d) (1 point) If we assume that n is large, then each slice is approximately a cylinder with height Δx and radius given by your answer to part (c). What is the volume of this cylinder? Leave your answer in terms of Δx and x_k.

> **Solution:**
> $$4\pi(5 - x_k)^2 \Delta x$$

(e) (2 points) Approximately how much work is needed to raise the water in the k^{th} slice to ground level?

> **Solution:** The water weights 1000 kg per meter cubed. We have $4\pi(5 - x_k)^2\Delta x$ meters cubed of water so
> $$4000\pi(5 - x_k)^2 \Delta x$$
> kg of water that we need to lift a distance of x_k meters. The force that needs to be applied is 10 times the mass so it is $40000\pi(5 - x_k)^2\Delta x$ Newtons. Using the fact that $W = Fd$ we get that the work done is equal to
> $$40000\pi x_k(5 - x_k)^2 \Delta x$$

(f) (3 points) Write a Riemann sum that calculates the amount of worked needed to drain the hole.

Solution:

$$\lim_{n\to\infty} \Delta x \sum_{k=1}^{n} 40000\pi x_k (5 - x_k)^2.$$

(g) (3 points) Use an integral to evaluate this sum.

Solution: We can convert the above sum to an integral and solve:

$$\begin{aligned} 40000\pi \int_0^5 x(5-x)^2 \, dx &= 40000\pi \left[\frac{25}{2}x^2 - \frac{10}{3}x^3 + \frac{1}{4}x^4 \right]_0^5 \\ &= 40000\pi 5^4 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \\ &= 40000 \cdot 5^4 \cdot \frac{1}{12} = \frac{10000\pi}{3} 5^4. \end{aligned}$$

2. (a) (3 points) Use long division to write the following fraction in the form $d(x) + \frac{r(x)}{q(x)}$ where the degree of $r(x)$ is less than the degree of $q(x)$.

$$\frac{p(x)}{q(x)} = \frac{x^4 - 2x^2 - 4x + 1}{x^3 + x^2 - x - 1}$$

Solution:

$$\begin{array}{r|l} & x - 1 \\ x^3 + x^2 - x - 1 & x^4 \qquad - 2x^2 - 4x + 1 \\ & -x^4 - x^3 + x^2 + x \\ \hline & \quad -x^3 - x^2 - 3x + 1 \\ & \quad\ \ x^3 + x^2 - x - 1 \\ \hline & \qquad\qquad -4x \end{array}$$

So

$$\frac{x^4 - 2x^2 - 4x + 1}{x^3 + x^2 - x - 1} = x - 1 - \frac{4x}{x^3 + x^2 - x - 1}$$

(b) (4 points) Calculate the integral $\int \frac{x^4 - 2x^2 - 4x + 1}{x^3 + x^2 - x - 1}\, dx$.

Solution: Using part (a) we can write the integral as

$$\int x - 1 - \frac{4x}{x^3 + x^2 - x - 1}\, dx.$$

We will use partial fractions to write

$$\frac{4x}{x^3 + x^2 - x - 1} = \frac{4x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Clearing denominators we get

$$4x = A(x+1)^2 + B(x^2 - 1) + C(x - 1)$$

so we see easily that $A = 1$ and $C = 2$. Looking at the coefficient of x^2 we get $0 = A + B$ and so $B = -1$. Thus we calculate

$$\int x - 1 - \frac{4x}{x^3 + x^2 - x - 1}\, dx = \frac{1}{2}x^2 - x + \ln|x - 1| - \ln|x + 1| - \frac{2}{x - 1} + C.$$

(c) (3 points) Convert the following Riemann sum into an integral of the form $\int_0^4 f(x)\,\mathrm{d}x$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\sqrt{1+32\frac{k^2}{n^2}}.$$

Solution: We have that $\Delta x = 4/n$ and $x_k = 4k/n$. Thus the sum can be written

$$\sum_{k=1}^{n}\frac{2}{n}\sqrt{1+32\frac{k^2}{n^2}} = \frac{4}{n}\sum_{k=1}^{n}\frac{1}{2}\sqrt{1+2\left(\frac{4k}{n}\right)^2}$$
$$= \Delta x\sum_{k=1}^{n}\frac{1}{2}\sqrt{1+2x_k^2}.$$

So we can easily see that the function is $f(x) = \frac{1}{2}\sqrt{1+2x^2}$.

(d) (4 points) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = te^t\sqrt{1+y}$ if $y(0) = -0.75$.

Solution: We start by separating variables and integrating

$$\int \frac{1}{\sqrt{1+y}}\,\mathrm{d}y = \int te^t\,\mathrm{d}t$$

We evaluate the right hand side using integration by parts. Let $u = t$ and $v' = e^t$. Then $u' = 1$ and $v = e^t$ so

$$\int te^t\,\mathrm{d}t = te^t - \int e^t\,\mathrm{d}t = te^t - e^t + C$$

The left hand side and be integraded using the power rule: $\int (1+y)^{-1/2}\,dy = 2\sqrt{1+y}$. Equating the two sides we get that

$$2\sqrt{1+y} = te^t - e^t + C.$$

We can now use the fact that $y(0) = -0.75$ to obtain $2\sqrt{1/4} = -1 + C$, i.e. $C = 2$ Thus plugging this in and rearranging,

$$y(t) = \frac{1}{4}(te^t - e^t + 2)^2 - 1.$$

3. Oxygen-15 is a radioactive isotope used in medicine. It has a half life of 120 seconds. During production, a solution containing 0.1 mg/ml of Oxygen-15 is added to a flask at a rate of 2 ml/s. Initially the flask contains 100 ml of pure water (i.e. initially there is no Oxygen-15 in the flask).

   (a) (1 point) How much Oxygen-15 (in mg) is being added, per second to the flask?

   Solution: 0.2 milligrams per second.

   (b) (2 points) Write a differential equation describing the total amount $y(t)$ of Oxygen-15 (in mg) in the flask at time t.

   Solution: We want to write a differential equation of the form $\frac{dy}{dt} = \text{rate in} - \text{rate out}$. The Oxygen-15 is entering at 0.2 milligrams per second. The only mechanism for it to leave the solution is via its half-life.

   Suppose for a moment that there is M milligrams of Oxygen-15. Then after t seconds the amount of Oxygen-120 is

$$\left(\frac{1}{2}\right)^{t/120} M = e^{-(t\ln 2)/120} M.$$

   By differentiating we see that, at time t, it is leaving at a rate of

$$-\frac{\ln 2}{120} e^{-(t\ln 2)/120} M = -\frac{\ln 2}{120}(\text{amount of O-15 at time } t)$$

   Hence the DE is

$$\frac{dy}{dt} = \frac{1}{5} - \frac{\ln 2}{120} y$$

(c) (2 points) Solve the differential equation from part (b).

Solution: Using separation of variables we arrive at the integrals

$$120 \int \frac{1}{24 - y \ln 2}\, dy = \int 1\, dt$$

The integral on the right is easy, the integral on the left can be done using the substitution $u = 24 - y \ln 2$, and we get

$$-\frac{120}{\ln 2} \int \frac{1}{u}\, du = -\frac{120}{\ln 2} \ln |u| = -\frac{120}{\ln 2} \ln |24 - y \ln 2| \, .$$

Putting everything together we get

$$-\frac{120}{\ln 2} \ln |24 - y \ln 2| = t + C$$

and rearranging

$$y(t) = \frac{24}{\ln 2} - Ce^{-\frac{\ln 2}{120} t}$$

Where C is an arbitrary constant. To find C we need to use the fact that $y(0) = 0$. We get the equation $0 = \frac{24}{\ln 2} - C$, so $C = \frac{24}{\ln 2}$ thus

$$y(t) = \frac{24}{\ln 2} \left(1 - e^{-\frac{\ln 2}{120} t}\right).$$

(d) (3 points) What is the concentration of Oxygen-15 (in mg/ml) in the flask at time t?

Solution: We want to find the number of milligrams per milliliter. We know there are $y(t)$ milligrams in total and the total amount of fluid is $100 + 2t$ millilitres (100 to start and 2 added every second). Thus the concentration is

$$\frac{y(t)}{100 + 2t} = \frac{24 \left(1 - e^{-\frac{\ln 2}{120} t}\right)}{\ln 2 (100 + 2t)}$$

(e) (5 points) Now suppose that, at the same time it is being filled, the flask is also being drained at a rate of 1 ml/s. Write a differential equation describing the total amount $y(t)$ of Oxygen-15 (in mg) in the flask at time t (you do not need to solve it).

Solution: We again look for a DE of the form $\frac{dy}{dt}$ = rate in − rate out. The rate in is still 0.2 and the Oxygen-15 is still leaving at a rate of $-\frac{\ln 2}{120}y$ due to the half-life. However now the Oxygen-15 is also leaving because it is being drained out.

At time t we have a concentration of $y(t)/(100+t)$ mg/ml. So in the 1 ml of fluid drained out, there is $y(t)/(100+t)$ mg of Oxygen-15. thus the DE is

$$\frac{dy}{dt} = \frac{1}{5} - \frac{\ln 2}{120}y - \frac{1}{100+t}y.$$

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