Midterm 1

UCLA: Math 3B, Fall 2016

Instructor: Noah White
Date: Monday, October 17, 2016
Version: *1.*

- This exam has 4 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: Solutions

ID number: ______________________

Discussion section: ______________________

Question	Points	Score
1	15	
2	8	
3	9	
4	8	
Total:	40	

1. Let $f(x) = \frac{x^2-9}{x^2+4}$. Note that $f'(x) = \frac{26x}{(x^2+4)^2}$ and $f''(x) = -\frac{26(3x^2-4)}{(x^2+4)^3}$.

   (a) (2 points) Find the x and y intercepts of $f(x)$.

   $f(0) = -9/4$ y-int: $y = -9/4$

   $f(x) = 0 \iff x = \pm 3$ x-int: $x = \pm 3$

   (b) (1 point) Does $f(x)$ have any horizontal asymptotes? If so what are they?

   $\lim_{x \to \pm\infty} f(x) = 1$ so yes @ $y = 1$

   (c) (1 point) Does $f(x)$ have any vertical asymptotes? If so what are they?

   yes @ $y = \pm 2$

   (d) (2 points) For what x is the first derivative $f'(x)$ positive?

   denominator always +ve

   so

f'	-ve	+ve
	$x < 0$	$x > 0$

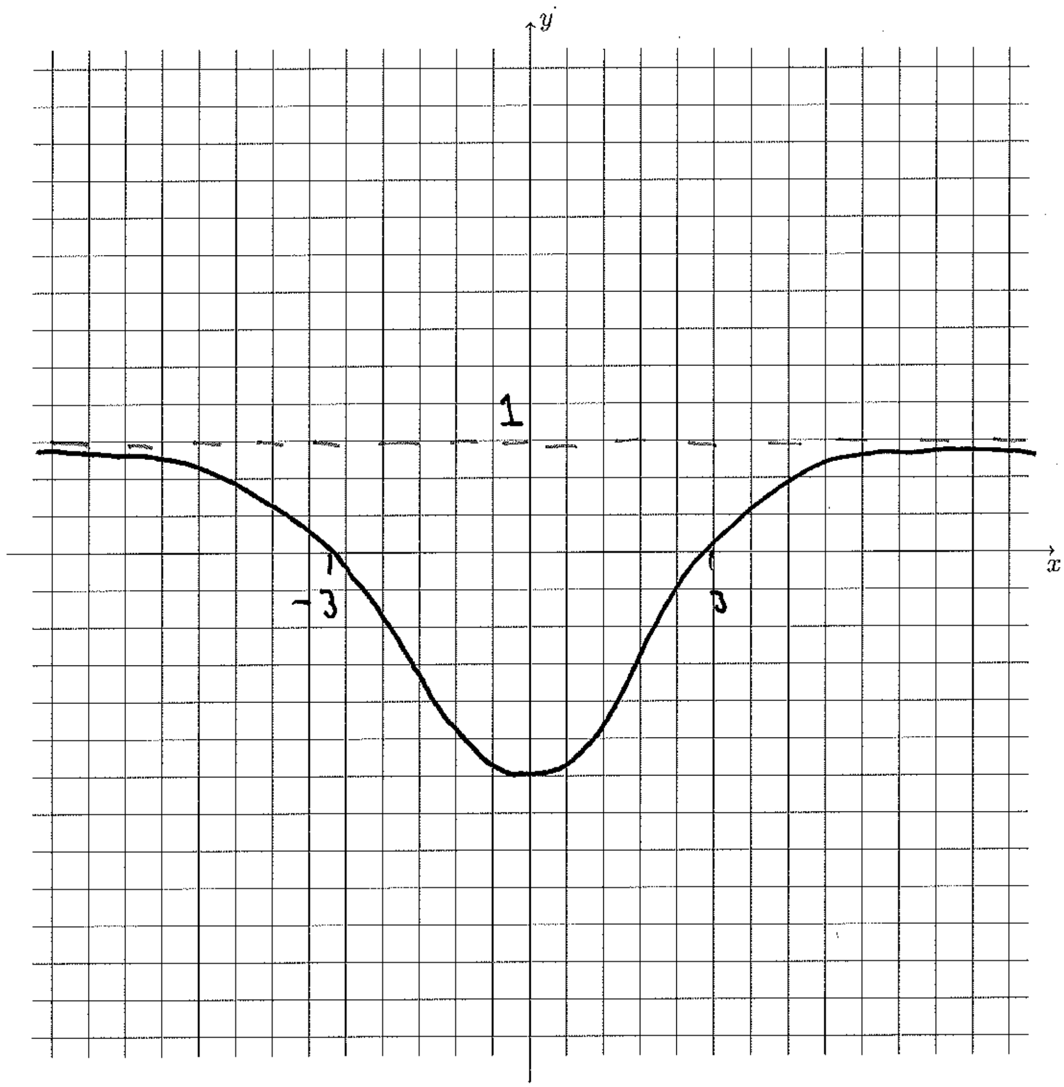
(e) (2 points) For what x is the second derivative $f''(x)$ positive?

denominator always +ve

so

f''	$-ve$	$+ve$
x	$\lvert x \rvert > \frac{2}{\sqrt{3}}$	$\lvert x \rvert < \frac{2}{\sqrt{3}}$

(f) (3 points) On the graph provided, sketch $f(x)$



(g) (4 points) List the local maximums and minimums of $f'(x)$ (note: this question if asking about the extrema of the derivative of f!)

* find critical points

ie when $(f')' = 0$ or undefined

$$0 = f''(x) = -\frac{26(3x^2-4)}{(x^2+4)^3}$$

$$\iff$$

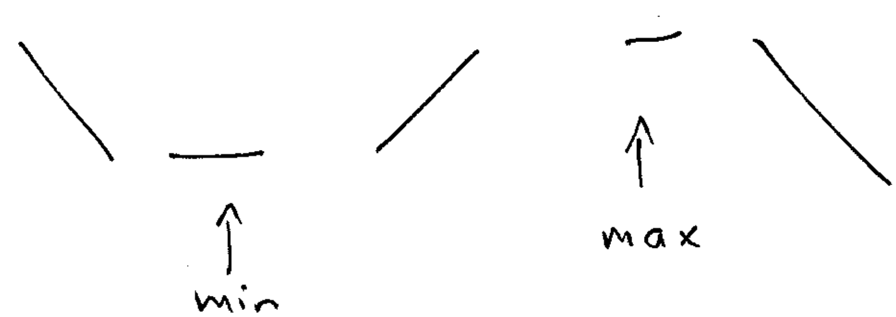
$3x^2 - 4 = 0$ ie $x = \pm \frac{2}{\sqrt{3}}$.

* $f''(x)$ neve undefined

* critical pnts: $x = \pm \frac{2}{\sqrt{3}}$

* from e):

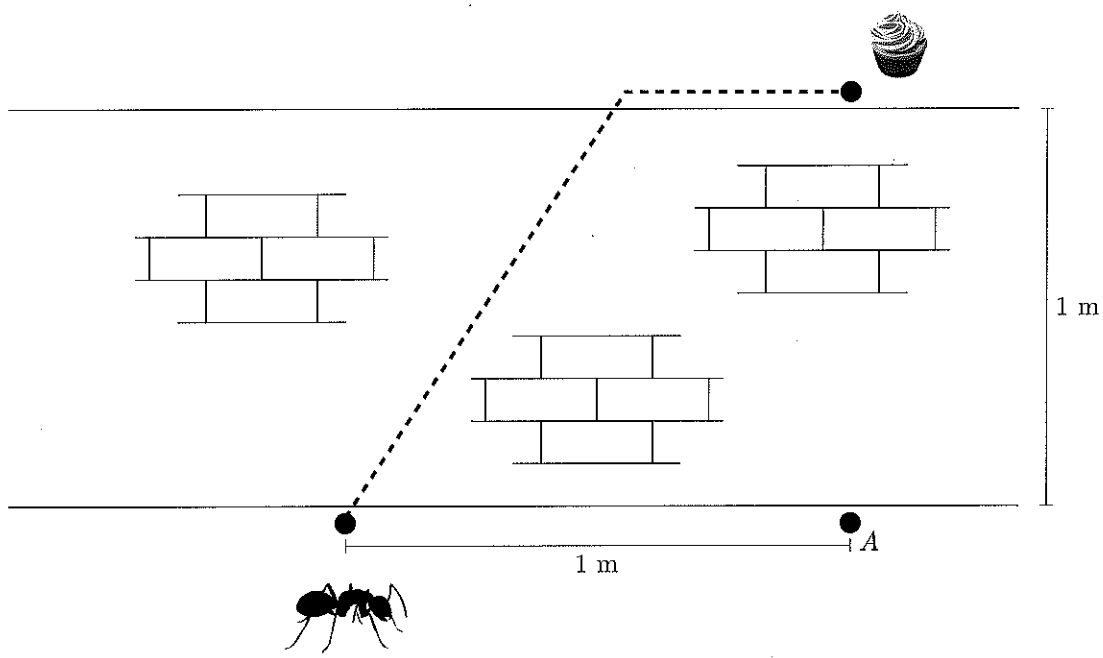
x	$x < -\frac{2}{\sqrt{3}}$	$x = -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < \frac{\sqrt{2}}{\sqrt{3}}$	$x = \frac{2}{\sqrt{3}}$	$x > \frac{2}{\sqrt{3}}$
f''	$-$	0	$+$	0	$-$



* $\therefore$ $x = -\frac{2}{\sqrt{3}}$ local min

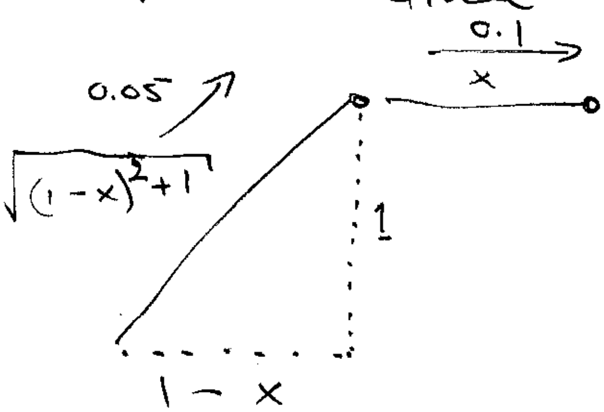
$x = 2/\sqrt{3}$ local max.

2. An ant wants to climb up a 1 meter high wall to reach a cupcake which is sitting on top of the wall directly above a point A on the ground 1 meter away from the ant. The ant can travel up the wall at a speed of 0.05m/s or along the top of the wall at a speed of 0.1m/s. It starts climbing the wall immediately.



(a) (3 points) If x is the distance between where the ant reaches the top of the wall and the cupcake, what is the time taken $T(x)$?

$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad \text{so} \quad \text{time} = \frac{\text{distance}}{\text{speed}}$$



so $$T(x) = \frac{x}{0.1} + \frac{1}{0.05}\sqrt{(1-x)^2+1}$$

$$= 10x + 20\sqrt{(1-x)^2+1}$$

(b) (5 points) How far from the cupcake should the ant reach the top of the wall in order to get to the cupcake in the quickest time possible?

* Need to minimise $T(x)$.

* Critical points when $T'(x) = 0$ / undef.

$$T'(x) = 10 + 20 \cdot \frac{1}{2} \frac{-2(1-x)}{\sqrt{(1-x)^2+1}}$$

$$= 10 \cancel{+} - \frac{20(1-x)}{\sqrt{(1-x)^2+1}}$$

$$= \frac{10\sqrt{(1-x)^2+1} - 20(1-x)}{\sqrt{(1-x)^2+1}}$$

* solve $T'(x) = 0$:

$$10\sqrt{(1-x)^2+1} = 20(1-x)$$

$$100(1-x)^2 + 100 = 400(1-2x+x^2)$$

$$200 - 200x + 100x^2 = 400 - 800x + 400x^2$$

$$0 = 200 \cancel{+} - 600x + 300x^2$$

$$0 = 2 \cancel{+} - 6x + 3x^2$$

$$x = 1 \pm \frac{1}{\sqrt{3}} \overset{\text{approx}}{=} 0.4 \quad \text{or} \quad 1.6 \leftarrow \text{too big.}$$

* fist der. test

	$x < 1-\frac{1}{\sqrt{3}}$	$x > 1-\frac{1}{\sqrt{3}}$
f'	$-$ve	$+$ve

$\therefore$ $\boxed{x = 1 - \frac{1}{\sqrt{3}}}$ is a minimum

3. Calculate the following integrals

(a) (3 points) $\int_0^1 x^2 + 3x - 4\,\mathrm{d}x$

$$= \left[\frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x \right]_0^1 = \left(\frac{1}{3} + \frac{3}{2} - 4 \right) - 0$$

$$= \frac{2 + 9 - 24}{6}$$

$$= -13/6 .$$

(b) (3 points) $\int x\cos(x^2 - 4)\,\mathrm{d}x$

$u = x^2 - 4$, $u' = 2x$ so

$$= \int \frac{1}{2}\cos(x^2-4)\,(2x)\,dx = \int \frac{1}{2}\cos(u)\,du$$

$$= \frac{1}{2}\{\sin u\} + C$$

$$= \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2-4) + C$$

(c) (3 points) $\int_1^e \frac{(\ln x)^2}{x}\,\mathrm{d}x$ $u = \ln x$ $u' = \frac{1}{x}$ $u(1) = 0,\ u(e) = 1$

$$= \int_1^e (\ln x)^2 \cdot \frac{1}{x}\,dx = \int_0^1 u^2\,du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3} .$$

4. A chemical manufacturer uses a reaction between two chemicals in solution, chemical A and chemical B to produce chemical X. Theoretically, it is understood that the reaction produces chemical X at a rate of

$$C_X(t) = \frac{1-t}{\sqrt{t+1}} \quad \text{ppm/s}$$

(parts per million per second), where a negative rate means the chemical X is being absorbed by the reaction and $t = 0$ is the time at the start of the reaction. Since chemicals A and B are very expensive, the manufacturer would like to stop the reaction only when the concentration of chemical X is at its highest.

(a) (1 point) At what time should the manufacturer stop the reaction?

Let $\mathcal{C}(t)$ be the concentration at time t. Want max of $\mathcal{C}(t)$. Note $\mathcal{C}'(t) = C_x(t)$. So $t = 1$ is a max of $\mathcal{C}(t)$. Should stop at $t = 1$.

(b) (5 points) Write a function that describes the concentration of chemical X at time t given that initially the solution initially contains 0ppm.

$u = \sqrt{t+1}$

so $t = u^2 - 1$

$u' = \frac{1}{2}\frac{1}{\sqrt{t+1}}$.

$$\mathcal{C}(t) = \int C_x(t)\,dt = \int \frac{1-t}{\sqrt{t+1}}\,dt$$

$$= 2\int (1-t)\left(\frac{1}{2}\frac{1}{\sqrt{t+1}}\right)dt$$

$$= 2\int 2 - u^2\,du$$

$$= 2\left(2u - \frac{1}{3}u^3\right) + C$$

$$= 4\sqrt{t+1} - \frac{2}{3}(t+1)^{3/2} + C$$

When $\mathcal{C}(0) = 0$ so

$$0 = 4\sqrt{1} - \frac{2}{3}(1)^{3/2} + C$$

$$0 = 4 - \frac{2}{3} + C$$

$$C = -\frac{10}{3}$$

$$\mathcal{C}(t) = 4\sqrt{t+1} - \frac{2}{3}(t+1)^{3/2} - \frac{10}{3}$$

$$= \frac{2}{3}\sqrt{t+1}\,(5-t) - \frac{10}{3}$$

(c) (2 points) What is the maximum yield of chemical X (in parts per million) the manufacturer can achieve?

Max yield achieved at $t = 1$.

$\cancel{G(t) - G(1)}$

$\cancel{G(1) = 2\sqrt{2}\left(\left(\frac{2}{3} - \frac{1}{3}\right)\right) - \frac{10}{3}}$

$\cancel{= 2\sqrt{2}\left(\frac{1}{3}\right) - \frac{10}{3}}$

$\cancel{= \frac{2}{3}\sqrt{2} - \frac{10}{3}}$

$\cancel{\approx}$

$$G(1) = \frac{2}{3}\sqrt{2}\,(4) - \frac{10}{3}$$

$$= \frac{8}{3}\sqrt{2} - \frac{10}{3}$$

$$\approx 0.44 \text{ ppm.}$$

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