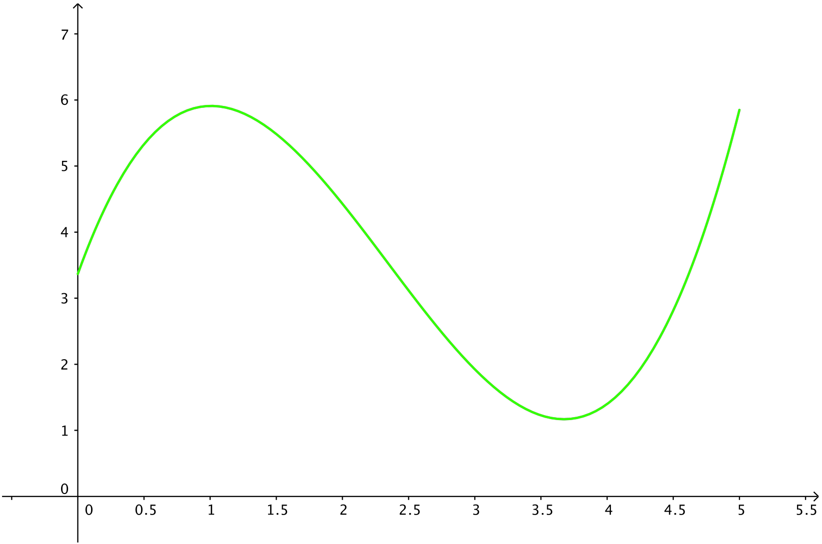
Math 3B: Lecture 8

Noah White

October 16, 2017

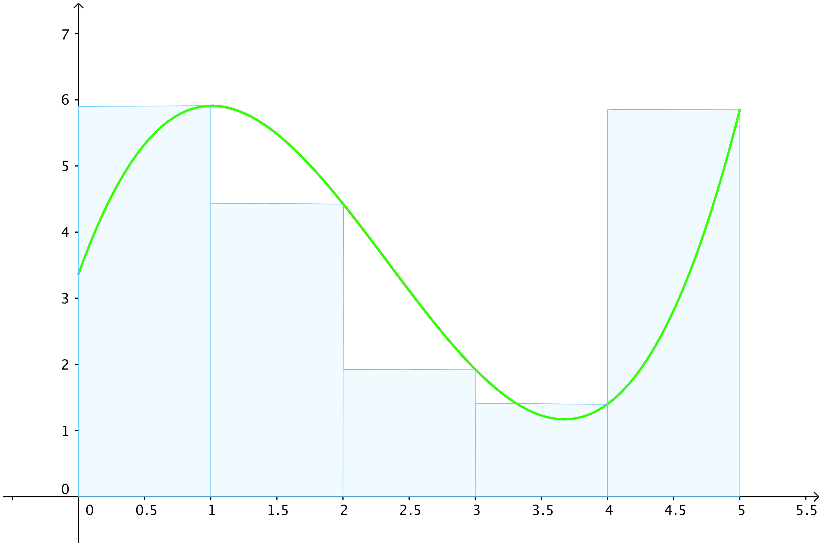
More complicated rates of change

Suppose we have a car whose speed is descibed by the following curve. How far has it travelled in this time?



More complicated rates of change

Suppose we have a car whose speed is descibed by the following curve. How far has it travelled in this time?



Accumulated change

- Suppose we know the rate of change $f(t)$, of some quantity (distance, water flow, population, etc).

Accumulated change

- Suppose we know the rate of change $f(t)$, of some quantity (distance, water flow, population, etc).
- How do we find the total amount by which this changes between $t = a$ and $t = b$?

Accumulated change

- Suppose we know the rate of change $f(t)$, of some quantity (distance, water flow, population, etc).
- How do we find the total amount by which this changes between $t = a$ and $t = b$?
- Answer: area under $f(t)$ between a and b.

Areas under general curves

We would like to calculate the area between a function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

Areas under general curves

We would like to calculate the area between a function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

A first approach to calculating the area under a curve is to approximate using rectangles:

Areas under general curves

We would like to calculate the area between a function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

A first approach to calculating the area under a curve is to approximate using rectangles:

(Too hard to draw, lets look at an animation)

The definite integral

Defintion

The definite integral of a function $f(x)$ is defined to be

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(a + k\Delta x)$$

where $\Delta x = \frac{b-a}{n}$.

The fundamental theorem of calculus

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

The fundamental theorem of calculus

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\, \mathrm{d}t = f(x)$$

The fundamental theorem of calculus

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

That is, $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of $f(x)$!

Note

The fundamental theorem of calculus

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

That is, $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of $f(x)$!

Note

- $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is a function of x.

The fundamental theorem of calculus

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

That is, $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of $f(x)$!

Note

- $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is a function of x.
- every input x produces a number as an output.

A consequence (corrollary)

Corollary

For any antiderivative $F(x)$ of $f(x)$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

A consequence (corrollary)

Corollary

For any antiderivative $F(x)$ of $f(x)$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Why?

Well $F(x) = \int_a^x f(t)\,\mathrm{d}t + C$ for some a and C. So

$$\begin{aligned} F(b) - F(a) &= \int_a^b f(t)\,\mathrm{d}t + C - \int_a^a f(t)\,\mathrm{d}t - C \\ &= \int_a^b f(t)\,\mathrm{d}t \end{aligned}$$

Example 1

Question

Evaluate the definite integral

$$\int_0^1 x^2 - 4 \, \mathrm{d}x$$

Example 1

Question

Evaluate the definite integral

$$\int_0^1 x^2 - 4 \, \mathrm{d}x$$

Solution

An antiderivative of $x^2 - 4$ is $\frac{1}{3}x^3 - 4x$ so

$$\begin{aligned} \int_0^1 x^2 - 4 \, \mathrm{d}x &= \frac{1}{3} \cdot 1^3 - 4 - \frac{1}{3} \cdot 0^3 + 4 \cdot 0 \\ &= \frac{1}{3} - 4 = -\frac{11}{3} \end{aligned}$$

Example 2

Question

Evaluate the definite integral

$$\int_0^{\pi} \sin x \, \mathrm{d}x$$

Example 2

Question

Evaluate the definite integral

$$\int_0^{\pi} \sin x \, \mathrm{d}x$$

Solution

An antiderivative of $\sin x$ is $-\cos x$ so

$$\begin{aligned} \int_0^{\pi} \sin x \, \mathrm{d}x &= -\cos \pi + \cos 0 \\ &= -(-1) + 1 = 2 \end{aligned}$$