Math 3B: Lecture 15

Noah White

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How to deal with rational functions?

How can we integrate something like

$$\int \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} \, \mathrm{d}x$$

or

$$\int \frac{x + 2}{x^3 - x} \, \mathrm{d}x?$$

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

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This is still not something we can integrate so we need to go further.

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots$$

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How do we reverse this process?

Answer: partial fractions

When the denominator is $(ax+b)(cx+d)\cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2)\cdots(a_n x + b_n)$$

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we can always find constants $A_1, A_2, \ldots, n$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \ldots \frac{A_n}{a_n x + b_n}$$

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Multiplying both sides by $(x-1)(x+1)$

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x+1} + \frac{B(x-1)(x+1)}{x-1} \\ &= A(x-1) + B(x+1) \\ &= (A+B)x + (B-A) \end{aligned}$$

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$$A + B = 0 \qquad \text{and} \qquad B - A = 1$$

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So

$$-2A = 1 \quad \text{hence} \quad A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$

Repeated factors

What if $q(x)$ contains repeated factors? E.g. if $q(x) = (x-1)^2$ or $q(x) = (x-1)(x+2)^3$?

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For every factor $(ax+b)^k$ in $q(x)$, the partial fraction expansion has terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}.$$

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Side note: integrating $\frac{1}{x}$.

Recall that

Fact

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

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Using substitution this gives the formula

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + C.$$

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Recall that if $k > 1$

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$$\int \frac{1}{x^k}\, \mathrm{d}x = -\frac{1}{(k-1)x^{k-1}} + C$$

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using polynomial long division.
2. Write $\frac{r(x)}{q(x)}$ as a sum of fractions of the form
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using partial fractions
3. Integrate all these pieces seperately.

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$$\frac{x^4 - 3x^2 + 3}{x^2 - 1} = x^2 - 2 + \frac{1}{x^2 - 1} = x^2 - 2 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

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So

$$I = \frac{1}{3}x^2 - 2x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

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$$\frac{x^3 - 2x^2 + 4x}{(x-1)^3} = 1 + \frac{x^2 + x + 1}{(x-1)^3} = 1 + \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3}$$

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So

$$I = x + \ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + C.$$

Differential equations (motivation)

An (ordinary) differential equation (or ODE) is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

$$x^2 y'' + xy' + x^2 y = 0$$

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The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

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And so on.

Integration

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Note

The right hand side of the equation does not have any y's.

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- draw solutions for many other ODEs
- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)
- understand how sensitive ODEs are to their parameters.

Initial value problems

- If we try to solve the differential equation

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we get (by integrating)

$$y(t) = t^3 + \cos t + C.$$

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- The extra piece of data we need is called an "initial value".
- E.g. $y(0) = 2$.
- Then we see that $y(0) = 1 + C$, so $C = 1$.

How to think about differential equations

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- If this is positive we go up, negative we go down!

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- as some other variable changes (usually time)
- We can't do this directly, but we can write down an ODE that y satisfies instead.

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- Number of deaths is proportional to the total number of people. So

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In real life we would determine b and d experimentally. Let $r = b - d$. the instinsic growth rate. So our model is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

and we know $N(0) = 100$.

Behaviour of solutions

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

Case 1: $r = 0$

The population never grows or shrinks, it always stays the same (so $N(t) = 100$ for all t).

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The population is increasing indefinitely.

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Case 2: $r > 0$
The population is increasing indefinitely.

Case 3: $r < 0$
The population is decreasing indefinitely.

Solution to a simple ODE

Theorem

For any constant a, if y is a solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay$$

then y is given by

$$y = Ce^{ax}$$

for some constant C.

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Next time

We will see why, but for now we can verify it is actually a solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} Ce^{ax} = C \frac{\mathrm{d}}{\mathrm{d}x} e^a x = Cae^{ax} = ay.$$

Back to example 1

We know our population model was governed by

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for some constant C. To detirmine C, we need one extra piece of information, $N(0) = 100$.

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$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b-d)N.$$

So the general solution is

$$N(t) = Ce^{(b-d)t}$$

for some constant C. To detirmine C, we need one extra piece of information, $N(0) = 100$.

$$100 = Ce^{(b-d)} \quad \text{so} \quad C = 100e^{(d-b)}.$$

Logistic growth

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$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= bN - (d + kN)N \\ &= (b - d - kN)N = (r - kN)N \\ &= r\left(1 - \frac{kN}{r}\right)N = r\left(1 - \frac{N}{K}\right)N\end{aligned}$$

Where $K = r/k$.

Logistic growth

The equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

is called the Logistic equation and K is the carrying capacity.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 1. $N(0) = 0$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains 0.

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Key takeaway

Both $N(t) = 0$ and $N(t) = K$ are solutions to the ODE. They are called equalibrium solutions.

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Case 4. $N(0) \geq K$

In this case N is initially decreasing but decreases slower and slower as it gets close to K.

Checking solutions

The most straighforward way of checking a function $y = f(x)$ is a solution to a differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x, y)$$

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Reminder: Implicit differentiation

If we have an equation relating variables y and x, e.g.

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$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) &= \frac{\mathrm{d}}{\mathrm{d}x}1 \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) &= 0 \\ 2x + 2y'y &= 0 \end{aligned}$$

Another example

Lets differentiate

$$3x + \cos y = xy$$

To do this we apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides:

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Note

We can rearrange this to get

$$y' = \frac{3 - y}{x + \sin y}$$

a differential equation. Whatever y is, as long as it obeys the above relation, it is a solution to this ODE!

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4. solve for y!

Examples

On the board. . .