Midterm 2

UCLA: Math 31B, Spring 2017

Instructor: Noah White
Date: 22 May 2017
Version: b

- This exam has 4 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: Solutions

ID number: ______________________

Discussion section (please circle):

Day/TA	Jeanine	William	Yuejiao
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	8	
2	10	
3	12	
4	10	
Total:	40	

Some facts you might find useful. You may use any of them throughout the exam.

$$\ln(ab) = \ln a + \ln b$$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln a^b = b \ln a$$

$$\ln x \leq x - 1 \text{ for all } x > 0.$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

1. Calculate the following limits using any technique you like.

(a) (5 points)

$$\lim_{x\to\infty}\left(2e^{-x}+1\right)^{e^x+1}$$

(b) (3 points)

$$\lim_{x\to-\infty} xe^x$$

a) $\lim_{x\to\infty} 2e^{-x}+1 = 1$, $\lim_{x\to\infty} e^x+1=\infty$ so indet form

let $L=\lim_{x\to\infty}(2e^{-x}+1)^{e^x+1}$, so

$$\ln L = \lim_{x\to\infty}(e^x+1)\ln(2e^{-x}+1)$$

$$=\lim_{x\to\infty}\frac{\ln(2e^{-x}+1)}{(e^x+1)^{-1}} \underset{(L'H)}{=} \lim_{x\to\infty}\frac{-2e^{-x}/(2e^{-x}+1)}{-e^x(e^x+1)^{-2}}$$

$$=\lim_{x\to\infty}\frac{2(e^x+1)^2}{e^{2x}(2e^{-x}+1)} = \lim_{x\to\infty}\frac{2(e^{2x}+2e^x+1)}{2e^x+e^{2x}} = 2$$

so $L=e^2$.

b) $\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty}\frac{x}{e^{-x}} \underset{(L'H)}{=} \lim_{x\to-\infty}\frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^{-x} = \underline{0}.$

2. For each of the following improper integrals say whether it converges or diverges. If the integral converges, you should say what value it converges to. *Hint: for one of these it may benefit you to look back at question 1.*

   (a) (5 points)

$$\int_{-\infty}^{0} (x+1)e^x \, dx$$

   (b) (5 points)

$$\int_{0}^{\pi/2} \cot x \, dx$$

a) $u = x+1 \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$\int_{-\infty}^{0} (x+1)e^x dx = \lim_{R\to-\infty} \int_R^0 (x+1)e^x dx = \lim_{R\to-\infty} \left[(x+1)e^x \Big|_R^0 - \int_R^0 e^x dx \right]$$

$$= \lim_{R\to-\infty} \left[(x+1)e^x \Big|_R^0 - e^x \Big|_R^0 \right]$$

converges.

$$= \lim_{R\to-\infty} \left[1-(R+1)e^R - 1 + e^R \right] = \lim_{R\to\infty} -Re^R = 0.$$

b) $\cot x = \frac{\cos x}{\sin x}$. $u = \sin x$, $du = \cos x \cdot dx$

$$\int_0^{\pi/2} \cot x \, dx = \lim_{R\to 0^+} \int_R^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx = \lim_{R\to 0^+} \int_{\sin R}^{1} \frac{du}{u} = \lim_{R\to 0^+} \left[\ln u \Big|_{\sin R}^{1} \right]$$

$$= \lim_{R\to 0^+} \left[-\ln \sin R \right] = -\ln \left(\lim_{R\to 0^+} \sin R \right)$$

$$= \infty$$

does not converge.

3. For each of the following series say whether it converges or diverges. You do NOT need to justify your answer.

   Please write your answer directly next to the question. Grading scheme: 0 points for wrong, 1 point for no response, 3 points for correct.

   (a) (3 points) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. ☒ converges ○ diverges

   (b) (3 points) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3-n-1}}{n^4}$. ☒ converges ○ diverges

   (c) (3 points) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n\sqrt{n}-2n+1}$. ○ converges ☒ diverges

   (d) (3 points) $\sum_{n=1}^{\infty} a_n$ where the sequence of partial sums $(S_N)_{N=1}^{\infty}$ are described by

   $$S_N = e^{\frac{2N}{N+1}}.$$

   ☒ converges ○ diverges

a) Alt. series test.

b) $\frac{\sqrt{n^3-n-1}}{n^4} \leq \frac{n^{3/2}}{n^4} = \frac{1}{n^{5/2}}$, comparison test.

c) $\frac{\sqrt{n}}{n\sqrt{n}-2n+1} \sim \frac{1}{n}$, use limit comp. test

d) $\sum a_n = \lim_{N\to\infty} S_n = \lim_{N\to\infty} e^{\frac{2N}{N+1}} = e^2$.

4. (a) (7 points) Does the series $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ converge or diverge? You should justify your answer.
Hint: use the definition of the infinite sum and properties of the logarithm.

(b) (3 points) Use the direct comparison test and the previous part to show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

a) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{N\to\infty} S_N$ where $S_N = \sum_{n=1}^{N} \ln\left(\frac{n+1}{n}\right)$

by def. But

$$S_N = \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ldots + \ln\left(\frac{N+1}{N}\right)$$

$$= \ln\left(\frac{2\cdot 3\cdot \ldots \cdot (N+1)}{1\cdot 2 \cdots N}\right) \qquad \left(\ln(a) + \ln(b) = \ln(ab)\right)$$

$$= \ln\left(\frac{(N+1)!}{N!}\right) = \ln(N+1)$$

so $\lim_{N\to\infty} S_N = \infty$ thus the series diverges.

b) We know (from formula sheet) $\ln(x) \le x-1$.

so $\ln\left(\frac{n+1}{n}\right) \le \frac{n+1}{n} - 1 = \frac{1}{n}$

since $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges, using the comp. test,

$\sum_{n=1}^{\infty} \frac{1}{n}$ must diverge too.