Midterm 1

UCLA: Math 31B, Spring 2017

Instructor: Noah White
Date: 24 April 2017
Version: a

- This exam has 4 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: Solutions

ID number: ______________________

Discussion section (please circle):

Day/TA	Jeanine	William	Yuejiao
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Some formulas you might find useful.

Ordinary trig functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$$

Hyperbolic trig functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh x = \operatorname{sech}^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{csch} x = -\operatorname{csch} x \coth x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\coth x = -\operatorname{csch}^2 x$$

Inverse trig functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1} x = -\frac{1}{1+x^2}$$

Inverse hyperbolic trig functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1} x = \frac{1}{1-x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sech}^{-1} x = \frac{1}{x\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{csch}^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\coth^{-1} x = \frac{1}{1-x^2}$$

1. (a) (4 points) Calculate

$$\int_e^a \frac{1}{2x\ln x}\,\mathrm{d}x$$

where $a = e^{e^2}$.

(b) (6 points) Calculate

$$\int_0^{\ln\sqrt{3}} \frac{e^x}{e^{2x}+1}\,\mathrm{d}x$$

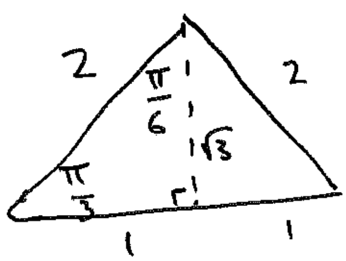
a) $u = \ln x \quad du = \frac{dx}{x}, \quad u(e) = 1, \quad u(e^{e^2}) = e^2$

$$\int_e^a \frac{1}{2x\ln x}\,dx = \int_1^{e^2} \frac{1}{2u}\,du$$

$$= \frac{1}{2}\ln u \Big|_1^{e^2} = \frac{1}{2}(2-0) = 1$$

b) $u = e^x, \quad du = e^x dx \quad u(0) = 1, \quad u(\ln\sqrt{3}) = \sqrt{3}$

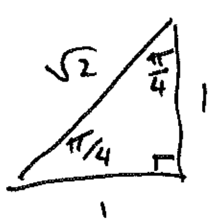
$$\int_0^{\ln\sqrt{3}} \frac{e^x}{e^{2x}+1}\,dx = \int_1^{\sqrt{3}} \frac{du}{u^2+1} = \tan^{-1} u\Big|_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4}$$



so $\tan\frac{\pi}{3} = \sqrt{3}$

$$= \frac{\pi}{12}$$

$\tan\frac{\pi}{4} = 1$



2. (a) (3 points) Let $f(x) = x^4 - x^3 + 4x^2 + 3x - 2$. What are the Taylor polynomials $T_2(x)$ and $T_4(x)$ of $f(x)$ centered at 0?

   (b) (2 points) Find the n^{th} Taylor polynomial about 0 of the function $\frac{1}{1+x}$.

   (c) (5 points) Let $T_n(x)$ be the n-th Taylor polynomial for $e^{\frac{1}{2}x}$ centered at 0. Find an n such that

$$\left|\sqrt{e} - T_n(1))\right| \leq \frac{1}{10^{113}}.$$

a) $T_2(x) = 4x^2 + 3x - 2$

$T_4(x) = f(x)$

b) $f^{(n)}(x) = (-1)^n \frac{n!}{(1+x)^{n+1}}$, $f^{(n)}(0) = (-1)^n n!$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} (-1)^k x^k$$

$$= 1 - x + x^2 - \ldots \pm x^n$$

c) $f(x) = e^{\frac{1}{2}x}$. $f^{(n)}(x) = \frac{e^{\frac{1}{2}x}}{2^n}$. We know that $|f^{(n+1)}(x)| \leq \frac{\sqrt{e}}{2^{n+1}}$ for all $0 \leq x \leq 1$, so by the error bound theorem:

$$|f(1) - T_n(1)| = |\sqrt{e} - T_n(1)| \leq \frac{\sqrt{e}}{2^{n+1}} \frac{1^{n+1}}{(n+1)!} = \frac{\sqrt{e}}{2^{n+1}(n+1)!}$$

In order to make sure that $\frac{\sqrt{e}}{2^{n+1}(n+1)!} \leq \frac{1}{10^{113}}$ we observe,

$$\frac{\sqrt{e}}{2^{n+1}(n+1)!} = \frac{\sqrt{e}}{\underbrace{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n+2)}_{n+1 \text{ factors}}} \leq \frac{1}{\underbrace{10 \cdot 12 \cdots (2n+2)}_{n-3 \text{ factors}}} \leq \frac{1}{10^{n-3}}$$

so we should make sure that $n - 3 \geq 113$

ie $n = 116$ will do.

3. (a) (6 points) Suppose $a > 0$. Calculate the following definite integral using a u-substitution.

$$\int \frac{1}{a^2 - x^2}\, \mathrm{d}x$$

(b) (2 points) Give a formula for $\tan(\sin^{-1} x)$ which does not involve trignometric functions.

(c) (2 points) Give a formula for $\tanh(\sinh^{-1} x)$ which does not involve (hyperbolic) trignometric functions.

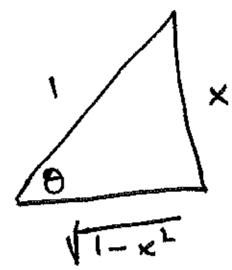
a) Note that $\dfrac{1}{a^2 - x^2} = \dfrac{1}{a^2} \cdot \dfrac{1}{1 - \left(\frac{x}{a}\right)^2}$

$u = \dfrac{x}{a}$, $du = \dfrac{dx}{a}$ so

$$\int \frac{dx}{a^2 - x^2} = \int \frac{1}{a^2} \frac{dx}{1 - \left(\frac{x}{a}\right)^2} = \frac{1}{a} \int \frac{du}{1 - u^2} = \frac{1}{a} \tanh^{-1} u + C$$

$$= \frac{1}{a} \tanh^{-1} \frac{x}{a} + C$$

b) let $\theta = \sin^{-1} x$, ie $x = \sin\theta$



so $\tan\theta = \tan(\sin^{-1} x)$

$$= \frac{x}{\sqrt{1 - x^2}}$$

c) let $\theta = \sinh^{-1} x$, so $x = \sinh\theta$

Now $\cosh^2\theta - \sinh^2\theta = 1$ so $\dfrac{1}{\tanh^2\theta} - 1 = \dfrac{1}{\sinh^2\theta}$

thus $(\tanh^2\theta)^{-1} = \dfrac{1}{\sinh^2\theta} + 1$

$$= \frac{1}{x^2} + 1 = \frac{1 + x^2}{x^2}$$

so $\tanh^2\theta = \dfrac{x^2}{1 + x^2}$ and

$$\tanh(\sinh^{-1} x) = \tanh(\theta) = \frac{x}{\sqrt{1 + x^2}}.$$

4. (10 points) Calculate the following indefinite integral

$$\int \frac{3e^{2x} - 10e^x + 5}{(e^x - 2)^2(e^x + 1)}\, dx.$$

First make the substitution $u = e^x$, $du = e^x dx$, $dx = \frac{du}{u}$

$\int \frac{3u^2 - 10u + 5}{(u-2)^2(u+1)u}\, du$. Now use partial fractions

$$\frac{3u^2 - 10u + 5}{(u-2)^2(u+1)u} = \frac{A}{u-2} + \frac{B}{(u-2)^2} + \frac{C}{u+1} + \frac{D}{u}$$

$$3u^2 - 10u + 5 = A(u-2)(u+1)u + B(u+1)u + C(u-2)^2u + D(u-2)^2(u+1)$$

$\underline{u=0}$: $5 = 4D$, $D = 4/5$ | $\underline{u=2}$: $-3 = 6B$

$\underline{u=-1}$: $18 = -9C$ $\quad C = -2$ | $B = -1/2$.

$\underline{u=1}$: $-2 = -2A + 2B + C + 2D$

$= -2A - 1 - 2 + 8/5$

$= -2A - 3 + 8/5$

$-2A = 1 - 8/5 = -3/5$

$A = 3/10$

$$\int \frac{3u^2 - 10u + 5}{(u-1)^2(u+1)u}\, du = \int \frac{3/10}{u-2} - \frac{1/2}{(u-2)^2} - \frac{2}{u+1} + \frac{4/5}{u}\, du$$

$$= \frac{3}{10}\ln|u-2| + \frac{1}{2(u-2)} - 2\ln|u+1|$$

$$+ \frac{4}{5}\ln|u| + C.$$