

This weeks problem set focuses on the ideas of linear combinations, linear dependence and bases. A question marked with a  $\dagger$  is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a  $*$  is especially important.

**Homework:** due Friday 19 Jan: questions 3 and 4 below.

1. From section 1.4, problems 1, 7, 8 ( $P_n(F)$  is the set of polynomials of degree less than or equal to  $n$ ), 11, 12, 13\*.
2. From section 1.5, problems 1, 2a, c, e, 4\*, 5, 9\*, 15, 18\*.
- 3\* Let  $V$  be a vector space over a field  $\mathbb{F}$  and  $W$  a subspace of  $V$ . For any  $v \in V$ , consider the set  $\{v\} + W = \{v + w \mid w \in W\}$ . We will denote it simply as  $v + W$ . Now consider the set

$$V/W = \{v + W \mid v \in V\}.$$

We can define addition and scalar multiplication on this set by

$$(v + W) + (w + W) = (v + w) + W \quad \text{and} \quad \lambda(v + W) = \lambda v + W.$$

Prove that  $V/W$  is a vector space. It is called the *quotient* of  $V$  by  $W$ .

4. Let  $\mathbb{C}[x]$  be the vector space of polynomials and let  $W = \text{span}\{x^a \mid a > 2\}$ .
  - (a) Find a set of 3 linearly independent elements of  $\mathbb{C}[x]/W$ .
  - (b) Find 2 nonzero elements  $p, q \in \mathbb{C}[x]$  that are linearly independent and such that  $p + W$  and  $q + W$  are linearly dependent and nonzero. *Note: you can only receive full points for this problem if your polynomials  $p$  and  $q$  are different from everyone else's! If you understand the problem then this will be easy to ensure. Please write your two polynomials very large on the top of your homework.*

**Note on problem 3:** The astute reader might be worried that the addition and scalar multiplication might not be well defined. What do I mean by this? Well, it is entirely possible that  $v + W = v' + W$  for two different elements  $v, v' \in V$ . This means we could calculate a sum in two different ways. As

$$(v + W) + (u + W) = (v + u) + W$$

or as

$$(v + W) + (u + W) = (v' + W) + (u + W) = (v' + u) + W$$

(since  $v + W = v' + W$ ). So we need to check that  $(v + u) + W = (v' + u) + W$ . I will show you how to do this below. You might like to try to prove that the scalar multiplication is unambiguous for yourself.

*Proof that  $(v + u) + W = (v' + u) + W$ :* Note that  $(v + u) + W = \{(v + u) + w \mid w \in W\}$  and  $(v' + u) + W = \{(v' + u) + w \mid w \in W\}$ . Also note that  $v \in v + W$  since  $v = v + 0$  and  $0 \in W$ .

Since  $v + W = v' + W$  we see that  $v \in v' + W$  and thus  $v = v' + x$  for some  $x \in W$ . Now let's take an arbitrary element  $s \in (v + u) + W$ , it will be of the form  $s = v + u + w$ . We know

$$s = v + u + w = v' + x + u + w = (v' + u) + (x + w).$$

Since  $x + u \in W$  we see that  $s = (v' + u) + (x + w) \in (v' + u) + W$ . We have just shown that  $(v + u) + W \subset (v' + u) + W$ . To complete the proof we need to show the opposite containment.

We do this in almost the same way. Take an arbitrary element  $t \in (v' + u) + W$ . We have that  $t = v' + u + w$  for some  $w \in W$ . Then

$$t = v' + u + w = v - x + u + w = (v + u) + (w - x) \in (v + u) + W.$$

Thus we have shown  $(v' + u) + W \subset (v + u) + W$  and hence  $(v + u) + W = (v' + u) + W$ .