Final practice

UCLA: Math 115A, Winter 2018

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Date:
Version: practice

- This exam has 8 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____________________

ID number: ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Question 2 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. In each of the following questions, fill in the blanks to complete the statement of the definition or theorem.

   (a) (2 points) *Definition:* Let V be a vector space. A subset B is called a basis of V if B is

   - ____________________________________ and
   - is a _______________ set.

   (b) (2 points) *Definition:* Let V and W be vector spaces over a field $\mathbb{F}$. A function $T : V \longrightarrow W$ is a linear map if

   - ____ for every $v, w \in V$, and
   - ____ for every $\lambda \in \mathbb{F}$ and $v \in V$.

   (c) (2 points) *Theorem:* Suppose $\dim V = n$. If $T : V \longrightarrow V$ is a linear map with _______________ distinct eigenvalues, then T is ______________________________.

   (d) (2 points) *Definition:* A basis B for a vector space V equipped with an inner product is called orthonormal if

   - ____________________________________ for every $v \in B$, and
   - ____ for every $v, w \in B$.

   (e) (2 points) *Definition:* The Frobenius inner product is an inner product defined on the vector space ____________________________________ and is defined by the formula

   ____.

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) The subset

$$V_\lambda = \{\, p \in \mathbb{R}[x] \mid p'(\lambda) = 0 \,\}$$

is a subspace

A. for any choice of $\lambda \in \mathbb{R}$.
B. only for $\lambda = 0$.
C. whenever $\lambda > 0$.
D. for no choice of λ.

(b) (2 points) As a subset of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$, the set

$$\left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\}$$

A. is a spanning set.
B. is linear independent.
C. is neither spanning nor linearly independent.
D. is a basis.

(c) (2 points) Suppose $n \geq 2$. Consider the function $T : \mathrm{Mat}_{n\times n}(\mathbb{C}) \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C})$ given by $T(M) = M^t$, the transpose matrix. Which of the following is *not* true.

A. T is a linear map.
B. $T^2 = -\,\mathrm{id}$.
C. T has eigenvalues 1 and -1.
D. T has an eigenspace of dimension $\frac{1}{2}n(n+1)$.

(d) (2 points) Consider again, the map T given above. Which of the following is true.

A. T is diagonalisable, only when $n = 2$.
B. T is diagonalisable for any n.
C. T is never diagonalisable.
D. The characteristic polynomial for T does not split.

(e) (2 points) Which of the following pairs of matrices are orthogonal in $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ equipped with the Frobenius inner product?

A. $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

B. $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$

C. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

D. $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

3. Consider the vector space $V = P_2(\mathbb{R})$ with its standard ordered basis

$$\beta = \{1, x, x^2\}$$

and the linear maps

$$T : P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ T(f) = f(1) + f(-1)x + f(0)x^2$$
$$S : P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ S(ax^2 + bx + c) = cx^2 + bx + a.$$

(a) (2 points) What is $[T]_\beta$ and $[S]_\beta$? Show that

$$[TS]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (6 points) Compute $[(TS)^{-1}]_\beta$.

(c) (2 points) What is $(TS)^{-1}(x^2 + x + 1)$?

4. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}),$$

   (a) (2 points) Compute the characteristic polynomial of A and determine the eigenvalues and their algebraic multiplicity.

   (b) (6 points) Is A is diagonalizable? If yes, compute a basis β of eigenvectors of A.

   (c) (2 points) Compute $[\mathrm{L}_A]_\beta$, where the L_A is the linear transformation given by

$$\mathrm{L}_A : \mathbb{R}^3 \to \mathbb{R}^3,\ v \mapsto Av.$$

5. Consider the vector space $V = \mathbb{R}^4$ with its standard inner product. Consider the linearly independent subset

$$S = \{w_1 = (1,0,1,0), w_2 = (1,1,1,1), w_3 = (2,2,0,2)\}.$$

   (a) (6 points) Apply the Gram-Schmidt orthogonalization algorithm to S to compute an orthogonal basis β' of $\mathrm{Span}(S)$.

   (b) (2 points) Use your result to compute an orthonormal basis β of $\mathrm{Span}(S)$.

   (c) (2 points) Let $x = (1,2,3,2) \in \mathrm{Span}(S)$. Compute the coordinate vector $[x]_\beta$.

6. Let $S : U \to V$ and $T : V \to W$ be linear transformations between finite dimensional vector spaces U, V and W over a field F.

   (a) (2 points) Let $v_1, v_2, \dots, v_n \in V$ be linearly independent and $\lambda_1, \lambda_2, \dots, \lambda_n \in F$, such that $\lambda_i \neq 0$ for all $1 \leq i \leq n$. Show that also $\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n$ are linearly independent.

   (b) (4 points) Let $v, w \in V$. Show that $\operatorname{span}(v, w) = \operatorname{span}(v, v + w)$.
   Hint: Proceed in two steps: Show that for all $x \in V$,

      1. $x \in \operatorname{span}(v, w)$ implies $x \in \operatorname{span}(v, v + w)$ and
      2. $x \in \operatorname{span}(v, v + w)$ implies $x \in \operatorname{span}(v, w)$.

   (c) (4 points) Assume that $\mathrm{R}(S) = \mathrm{N}(T)$, i.e. the range of S is equal to the nullspace of T. Assume furthermore that S is one-to-one and T is onto. Show that

   $$\dim V = \dim U + \dim W.$$

   Hint: Use the rank-nullity formula.

7. Let V be a finite dimensional vector space over a field F. Recall that

$$\mathcal{L}(V, V) = \{T : V \to V \mid T \text{ is a linear transformation}\}$$

denotes the vector space of linear transformations from V to V (also called linear operators on V). Fix a vector $v \in V$ and define

$$Z = \{T \in \mathcal{L}(V, V) \mid T(v) = 0\}\,.$$

One calls Z the *annihilator* of v in $\mathcal{L}(V, V)$.

(a) (4 points) Show that Z is a subspace of $\mathcal{L}(V, V)$.

(b) (2 points) Let $\lambda \in F$ such that $\lambda \neq 0$. Prove or disprove (by finding a counterexample) that

$$Z' = \{T \in \mathcal{L}(V, V) \mid T(v) = \lambda v\}$$

is a subspace of $\mathcal{L}(V, V)$.

(c) (2 points) Assume that $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of V, such that $v_1 = v$. Let $T \in \mathcal{L}(V, V)$. Show that $T \in Z$ if and only if the first column of $A = [T]_\beta$ equals 0.

(d) (2 points) Assuming $v \neq 0$, what is $\dim(Z)$?

8. Let V be a finite dimensional vector space over $\mathbb{R}$ with an inner product $\langle x, y\rangle \in \mathbb{R}$ for $x, y \in V$.

   (a) (3 points) Let $\lambda \in \mathbb{R}$ with $\lambda > 0$. Show that

   $$\langle x, y\rangle' = \lambda \langle x, y\rangle, \text{ for } x, y \in V,$$

   defines an inner product on V.

   (b) (2 points) Let $T : V \to V$ be a linear operator, such that

   $$\langle T(x), T(y)\rangle = \langle x, y\rangle, \text{ for all } x, y \in V.$$

   Show that T is one-to-one.

   (c) (2 points) Recall that the *norm* of a vector $x \in V$ is defined by $||x|| = \sqrt{\langle x, x\rangle}$. Show that

   $$\langle x, y\rangle = \frac{1}{2}(||x + y||^2 - ||x||^2 - ||y||^2), \text{ for all } x, y \in V.$$

   Hence, the inner product can be recovered from the norm.
   Hint: Rewrite $\langle x + y, x + y\rangle$ using the properties of inner products. Use that $\langle x, y\rangle \in \mathbb{R}$ is a real number by assumption.

   (d) (3 points) Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V. The *Gram matrix* $G \in \mathrm{M}_{n,n}(\mathbb{R})$ of the inner product $\langle -, -\rangle$ with respect to β is defined by

   $$G_{i,j} = \langle v_i, v_j\rangle.$$

   Show that G is invertible.

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