MATH 131A Summer 2024

Number systems

Integers

The **integers** are a set $\mathbb{Z}$ with an operation $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, called **addition**, and an operation $\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, called **multiplication**, that satisfy the following properties.

For all $a, b, c \in \mathbb{Z}$,

(A1) $(a + b) + c = a + (b + c)$ [**associativity** of addition],

(A2) $a + b = b + a$ [**commutativity** of addition],

(A3) there exists an element $0 \in \mathbb{Z}$ such that $a + 0 = a$ [existence of an additive **identity**],

(A4) there exists an element $-a \in \mathbb{Z}$ such that $a + (-a) = 0$ [existence of additive **inverses**],

(M1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ [**associativity** of multiplication],

(M2) $a \cdot b = b \cdot a$ [**commutativity** of multiplication],

(M3) there exists an element $1 \in \mathbb{Z} \setminus \{0\}$ such that $a \cdot 1 = a$ [existence of a multiplicative **identity**],

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ [**distributivity** of multiplication over addition],

(Z) if $a \cdot b = 0$, then $a = 0$ or $b = 0$ [no nonzero zero divisors].

The **natural numbers** are a set $\mathbb{N} \subseteq \mathbb{Z}$ that satisfy the following properties.

(N1) $\mathbb{N} + \mathbb{N} \subseteq \mathbb{N}$ [**closure** under addition],

(N2) $\mathbb{N} \cdot \mathbb{N} \subseteq \mathbb{N}$ [**closure** under multiplication],

(N3) $\mathbb{N} \cap -\mathbb{N} = \{0\}$ [intersection with its negative is $\{0\}$],

(N4) $\mathbb{N} \cup -\mathbb{N} = \mathbb{Z}$ [union with its negative is $\mathbb{Z}$].

(Here $\mathbb{N} + \mathbb{N} := \{a + b : a \in \mathbb{N} \wedge b \in \mathbb{N}\}$, $\mathbb{N} \cdot \mathbb{N} := \{a \cdot b : a \in \mathbb{N} \wedge b \in \mathbb{N}\}$, and $-\mathbb{N} := \{-a : a \in \mathbb{N}\}$.)

Order

If we define the relation $\leq$ on $\mathbb{Z}$ as $\{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b - a \in \mathbb{N}\}$ (that is, $a \leq b$ if and only if $b - a \in \mathbb{N}$), then $\leq$ is a **total order** on $\mathbb{Z}$. That is, for all $a, b, c \in \mathbb{Z}$,

- $a \leq a$ [the relation is **reflexive**],
- if $a \leq b$ and $b \leq a$, then $a = b$ [the relation is **antisymmetric**],
- if $a \leq b$ and $b \leq c$, then $a \leq c$ [the relation is **transitive**],

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- $a \leq b$ or $b \leq a$ [the relation is **strongly connected**].

In addition,

- if $a \leq b$, then $a + c \leq b + c$,
- if $0 \leq a$ and $0 \leq b$, then $0 \leq a \cdot b$.

Finally, we have the following property:

(W) if $S \subseteq \mathbb{N}$ and $S \neq \varnothing$, then there exists an $\ell \in S$ such that $\ell \leq s$ for all $s \in S$ [the relation is a **well-order** on $\mathbb{N}$].

Absolute value

The **absolute value** on $\mathbb{Z}$ is the function $|\cdot| : \mathbb{Z} \to \mathbb{Z}$ given by

$$|a| := \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{otherwise.} \end{cases}$$

It has the following properties for all $a, b \in \mathbb{Z}$.

- $|a| \geq 0$ [**nonnegativity**],
- if $|a| = 0$, then $a = 0$ [**positive definiteness**],
- $|a \cdot b| = |a| \cdot |b|$ [**multiplicativity**],
- $|a + b| \leq |a| + |b|$ [**subadditivity/triangle inequality**].

Rational numbers

Let $\sim$ be the relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ such that $(a, b) \sim (c, d)$ if and only if $a \cdot d = b \cdot c$. The **rational numbers** are the set $\mathbb{Q}$ of equivalence classes of this relation.

The operations $+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ and $\cdot : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by

$$\begin{aligned} [(a, b)] + [(c, d)] &:= [(a \cdot d + b \cdot c, b \cdot d)], \\ [(a, b)] \cdot [(c, d)] &:= [(a \cdot c, b \cdot d)], \end{aligned}$$

are well-defined and obey axioms (A1)–(A4), (M1)–(M3), and (D) (with $\mathbb{Q}$ in place of $\mathbb{Z}$). Moreover, we have

(M4) for all $q \in \mathbb{Q} \setminus \{0\}$, there exists an element $r \in \mathbb{Q}$ such that $q \cdot r = 1$ [existence of multiplicative **inverses**].

Absolute value and order

The **absolute value** on $\mathbb{Q}$ is the function $|\cdot| : \mathbb{Q} \to \mathbb{Q}$ given by

$$|[(a, b)]| := [(|a|, |b|)].$$

It is also well-defined, and the **nonnegative rational numbers** defined as $\mathbb{Q}_{\geq 0} := \{q \in \mathbb{Q} : |q| = q\}$ satisfy axioms (N1)–(N4) (with $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}$ in place of $\mathbb{N}$ and $\mathbb{Z}$). The order relation $\leq$ on $\mathbb{Q}$ is then defined as $\{(q, r) \in \mathbb{Q} \times \mathbb{Q} : r - q \in \mathbb{Q}_{\geq 0}\}$.

Inclusion of integers

The map $\iota : \mathbb{Z} \to \mathbb{Q}$ given by

$$\iota(a) := [(a, 1)]$$

satisfies

- $\iota(a + b) = \iota(a) + \iota(b)$,
- $\iota(a \cdot b) = \iota(a) \cdot \iota(b)$,
- $a \leq b$ if and only if $\iota(a) \leq \iota(b)$,
- $\iota(|a|) = |\iota(a)|$,

for all $a, b \in \mathbb{Z}$. Thus, by identifying $\mathbb{Z}$ with $\iota(\mathbb{Z})$, we may regard $\mathbb{Z}$ as a subset of $\mathbb{Q}$.

Real numbers

Let $\sim$ be the relation on Cauchy sequences in $\mathbb{Q}$ such that $(q_n) \sim (r_n)$ if and only if $q_n - r_n \to 0$. The **real numbers** are the set $\mathbb{R}$ of equivalence classes of this relation.

The operations $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$[(q_n)] + [(r_n)] := [(q_n + r_n)],$$
$$[(q_n)] \cdot [(r_n)] := [(q_n \cdot r_n)],$$

are well-defined and obey axioms (A1)–(A4), (M1)–(M4), and (D) (with $\mathbb{R}$ in place of $\mathbb{Z}$ or $\mathbb{Q}$).

Absolute value and order

The **absolute value** on $\mathbb{R}$ is the function $|\cdot| : \mathbb{R} \to \mathbb{R}$ given by

$$|[(q_n)]| := [(|q_n|)].$$

It is also well-defined, and the **nonnegative real numbers** defined as $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : |x| = x\}$ satisfy axioms (N1)–(N4) (with $\mathbb{R}_{\geq 0}$ and $\mathbb{R}$ in place of $\mathbb{N}$ and $\mathbb{Z}$). The order relation $\leq$ on $\mathbb{R}$ is then defined as $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y - x \in \mathbb{R}_{\geq 0}\}$.

Inclusion of rational numbers

The map $\iota : \mathbb{Q} \to \mathbb{R}$ given by

$$\iota(q) := [(q)_{n=1}^{\infty}]$$

satisfies

- $\iota(q + r) = \iota(q) + \iota(r)$,

- $\iota(q \cdot r) = \iota(q) \cdot \iota(r)$,
- $q \leq r$ if and only if $\iota(q) \leq \iota(r)$,
- $\iota(|q|) = |\iota(q)|$,

for all $q, r \in \mathbb{Q}$. Thus, by identifying $\mathbb{Q}$ with $\iota(\mathbb{Q})$, we may regard $\mathbb{Q}$ as a subset of $\mathbb{R}$.