$P = \{x_i\}_{i=0}^{n}$ portion of $[a,b]$

$f : [a,b] \to \mathbb{R}$ bd. $x_i - x_{i-1}$

$U(f,P) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x]} f(x) \underbrace{\Delta x_i}_{x_i - x_{i-1}}$ $\qquad L(f,P) = \sum_{i=1}^{n} \inf \ldots$

$\overline{\int_a^b} f(x)\,dx = \inf_P U(f,P)$ $\qquad \underline{\int_a^b} f(x)\,dx = \sup_P L(f,P)$

If $P \subseteq P'$, then

$U(f,P) \geq U(f,P')$ $\qquad \limsup_n x_n = \inf_{n>1} \sup_{m>n} x_m$ $\qquad L(f,P) \leq L(f,P')$

In other words,

$\overline{\int_a^b} f(x)\,dx = \inf_P \sup_{P' \geq P} U(f,P')$ $\qquad \underline{\int_a^b} f(x)\,dx = \sup_P \inf_{P' \geq P} L(f,P')$

$\liminf_n x_n = \sup_{n>1} \inf_{m>n} x_m$

$\Downarrow$

$\sup_{P' \geq P} U(f,P') \leq U(f,P)$

and $P \geq P$.

So $\sup_{P' \geq P} U(f,P') = U(f,P)$

$\ast$ integratbility means

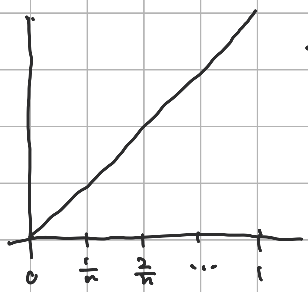
$\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$

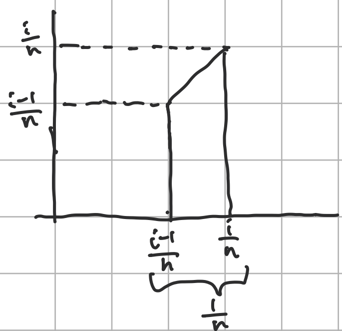
$\Vert \qquad \Vert$

$\int_a^b f\,dx$

<u>thm</u> f is int. on $[a,b]$ iff for all $\varepsilon > 0$, there exists a part. P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

e.x) $f(x) = x$





Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ (i.e., $x_i = \frac{i}{n}$ for $0 \leq i \leq n$)

Then $U(f,P_n) = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \cdot \frac{n(n+1)}{2}$

$= \frac{n^2+n}{2n^2} = \frac{1}{2} + \frac{1}{2n}$

$L(f,P_n) = \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n}$

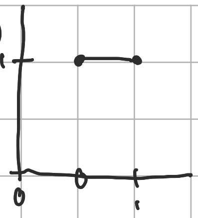
$= \frac{1}{n^2}\left(\frac{n(n+1)}{2} - n\right) = \frac{1}{2} - \frac{1}{2n}$

So $U(f,P_n) - L(f,P_n) = \frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$

Moreover

$L(f,P_n) \leq \underline{\int_0^1} x\,dx \overset{\text{INT}}{=} \overline{\int_0^1} x\,dx \leq U(f,P_n)$

Hence $\int_0^1 x\,dx = \frac{1}{2}$

e.x)  $f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$ Show f int. and compute $\int_0^1 f\,dx$.

e.x) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ for $x \in [0,1]$

Let P be part of $[0,1]$. Then $U(f,P) = \sum_{i=1}^{n} \overbrace{\sup_{x \in [x_{i-1}, x_i]} f(x)}^{=1 \text{ b/c}} \Delta x_i = 1$

$L(f,P) = 0$ b/c $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

So $U(f,P) - L(f,P) = 1 - 0$ for all P

$= 1$

Hence f is not integrable.

thm f is int. on $[a,b]$ w/ $\int_a^b f(x)\,dx = I$ iff for all $\varepsilon > 0$, there exists a partition P of $[a,b]$ s.t. $|I - U(f,P)| < \varepsilon$ and $|I - L(f,P)| < \varepsilon$. also true for all $P' \supseteq P$ b/c L is inc. and U is dec.

Equivalently $I - \varepsilon < L(f,P) \le U(f,P) < I + \varepsilon$

pf Suppose $\int_a^b f\,dx = I$.

Given an $\varepsilon > 0$, there exists a part. P_L s.t. $I - \varepsilon < L(f, P_L)$.

Similarly, there exists a part. P_U s.t. $U(f, P_U) < I + \varepsilon$.

Hence

$$I - \varepsilon < L(f, P_L) \le L(f, P_L \cup P_U) \le U(f, P_L \cup P_U) \le U(f, P_U) < I + \varepsilon$$

Conversely, given an $\varepsilon > 0$, there exists a part. P s.t.

$|I - U(f,P)| < \frac{\varepsilon}{2}$ and $|I - L(f,P)| < \frac{\varepsilon}{2}$, so

$U(f,P) - L(f,P) = |U(f,P) - L(f,P)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so $\int_a^b f\,dx$ exists.

Moreover, $I - \frac{\varepsilon}{2} < L(f,P) \le \int_a^b f\,dx \le U(f,P) < I + \frac{\varepsilon}{2}$, so $\int_a^b f\,dx = I$

thm If $f \in C([a,b])$, then f is int. on $[a,b]$.

idea Note that $U(f,P) - L(f,P) = \sum_{i=1}^{n} \overbrace{\left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right)}^{\text{small if } \Delta x_i \text{ is small}} \Delta x_i$

$= \text{small} \cdot \sum_i \Delta x_i = \text{small}\,(b-a)$

pf Given an $\varepsilon > 0$, there exists a $\delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \text{whenever} \quad |x-y| < \delta \quad (x, y \in [a,b])$$

since f is uniformly continuous.

Now let $P = \{a,\ a + \frac{b-a}{n},\ a + 2\cdot\frac{b-a}{n},\ \dots,\ a + n\cdot\frac{b-a}{n} = b\}$ with $\Delta x_i = \frac{b-a}{n} < \delta$. Then

$$U(f,P) - L(f,P) \le \sum_{i=1}^{n} \sup_{x,y \in [x_{i-1}, x_i]} |f(x) - f(y)|\, \Delta x_i \le \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \Delta x_i < \varepsilon.$$

$*$ $f(x) - f(y) \le \sup_{x,y} |f(x) - f(y)|$

$\Rightarrow \sup_{x} f(x) - \inf_{y} f(y) \le \sup_{x,y} |f(x) - f(y)|$

prop Let f, g be int. on $[a,b]$. Then

(a) $\int_a^b f + g\, dx = \int_a^b f\, dx + \int_a^b g\, dx$

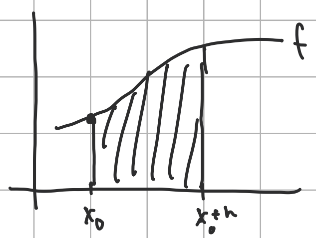
(b) $\int_a^b c f\, dx = c \int_a^b f\, dx$ for $\forall c \in \mathbb{R}$.

(c) if $f \le g$, then $\int_a^b f\, dx \le \int_a^b g\, dx$

(d) if $c \in (a,b)$, then f is int. on $[a,c]$ with $\int_a^b f\, dx = \int_a^c f\, dx + \int_c^b f\, dx$ and conversely

(e) $\left| \int_a^b f\, dx \right| \le \sup_{x \in [a,b]} |f(x)| \cdot (b-a)$

thm (FTC I) Let f be int. on $[a,b]$ and $F(x) := \int_a^x f(t)\, dt$ for $x \in [a,b]$. Then F is uniformly cont. on $[a,b]$ and if f is cont. at $x_0 \in [a,b]$, then $F'(x_0) = f(x_0)$.



$F(x_0 + h) - F(x_0)$
$\approx f(x_0) \cdot h$
so we should have

$$\underbrace{\lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h}}_{\overset{\shortparallel}{F'(x_0)}} = f(x_0)$$

pf For $a \le x \le y \le b$, we have

$$|F(y) - F(x)| = \left| \int_x^y f(t)\, dt \right| \le \overbrace{\sup_{x \in [a,b]} |f(x)|}^{\text{const.}\ =: M} \cdot |y - x| < \varepsilon$$

if $|y - x| < \frac{\varepsilon}{M}$, so F is unif. cont. on $[a,b]$.

If f is cont. @ x_0 and $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $x_0 \in [a,b]$ and $|t - x| < \delta$ then $|f(t) - f(x_0)| < \varepsilon$.

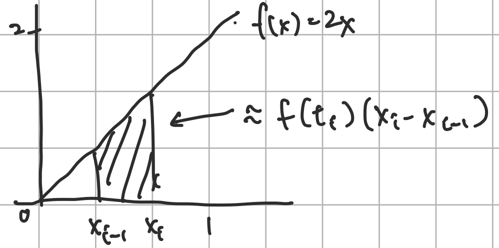
Hence for $0 < h < \delta$ s.t. $x_0 + h \in [a,b]$, we have

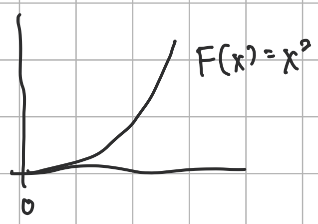
$$\left| \frac{F(x_0+h) - F(x_0)}{h} - \underbrace{f(x_0)}_{\text{const.}} \right| = \left| \frac{1}{h} \int_{x_0}^{x_0+h} f(t) - f(x_0)\, dt \right| \le \frac{1}{h} \varepsilon \cdot h = \varepsilon$$

We can argue similarly for $-\delta > h < 0$ s.t. $x_0 - h \in [a, b]$.

Thus $\lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h} = f(x_0)$

thm (FTC II) Let f be cnt. on $[a, b]$ and suppose F is an ANTIDERI. of f on (a, b), meaning $F' = f$.
Then $\int_a^b f(x)\,dx = F(b) - F(a)$.





$F = f'$ $F(x_i) - F(x_{i-1})$