5 Sep 02 – Sep 08	Differentiation • The derivative • Mean value theorem • Taylor's theorem [Ros13, ss. 28–29, 31; Rud76, pp. 103–108, 110–111]	PR 4 due (T, R) Quiz 5 (W) HW 4 due (F)

09.03 OH 13:30, MS 6147. Practice/past exams and midterm grades to be released today.

The Derivative

Def: Let $U \subseteq \mathbb{R}$ be open. The **derivative** of $f : U \to \mathbb{R}$ at $x \in U$ is

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(if the limit exists).}$$

If $f'(x)$ exists, we say that f is **differentiable** at x; if f is differentiable at every $x \in U$, we say that f is differentiable on U.

Rmk: We will also consider functions defined on closed intervals $[a, b]$ $(a < b)$, in which case the limits defining $f'(a)$ and $f'(b)$ only involve points on one side of a or b.

Prop: If f is differentiable at x, then it is continuous at x.

Pf:

$$f(x+h) - f(x) = \frac{f(x+h) - f(x)}{h} \cdot h \quad \underset{h \to 0}{\longrightarrow} f'(x) \cdot 0 = 0.$$

Notation:

$$C(U) = C^0(U) \text{ is the set of all continuous functions on } U.$$

$C^k(U)$ $(k \geq 1)$ is the set of all differentiable functions on U with derivative in $C^{k-1}(U)$.

Ex: $C^1(U) \subseteq C^0(U), \quad C^2(U) \subseteq C^1(U), \quad \text{etc.} \quad C^k(U) \subseteq C^{k-1}(U).$

For example:

$$f(x) = x^2, \quad f'(x) = 2x \quad \text{so} \quad f \in C^1(U).$$

$$f''(x) = 2 \quad \text{so} \quad f \in C^2(U).$$

In fact, $f \in C^k(U)$ for all $k \geq 0$.

Prop: If f and g are differentiable at x, then:

(a) if $f(x) \equiv c$ for some $c \in \mathbb{R}$, then $f'(x) \equiv 0$.

(b) $(f+g)'(x) = f'(x) + g'(x)$.

(c) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

(d) $\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{g(x)^2}$ if $g(x) \neq 0$.

Proof:

(a) Trivial.

(b)

$$\frac{(f+g)(x+h) - (f+g)(x)}{h} = \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \longrightarrow f'(x) + g'(x).$$

(c) The derivative of a function $h(x)$ at a point x is defined as:

$$h'(x) = \lim_{h\to 0} \frac{h(x+h) - h(x)}{h}$$

We will use this definition to compute the derivative of the product $f(x)g(x)$. Let $h(x) = f(x)g(x)$. The derivative of $h(x)$ at x is:

$$h'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$h'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}$$

Now the expression in the numerator is split into two parts:

$$h'(x) = \lim_{\Delta x\to 0} \left[\frac{f(x+\Delta x)\big(g(x+\Delta x) - g(x)\big)}{\Delta x} + \frac{\big(f(x+\Delta x) - f(x)\big)g(x)}{\Delta x} \right]$$

First part:

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)\big[g(x+\Delta x) - g(x)\big]}{\Delta x} = f(x) \cdot g'(x)$$

As $\Delta x \to 0$, $f(x+\Delta x) \to f(x)$ and the expression $\frac{g(x+\Delta x)-g(x)}{\Delta x} \to g'(x)$.
Second part:

$$\lim_{\Delta x\to 0} \frac{\big[f(x+\Delta x) - f(x)\big]g(x)}{\Delta x} = f'(x) \cdot g(x)$$

because $\frac{f(x+\Delta x)-f(x)}{\Delta x} \to f'(x)$, and $g(x)$ is constant with respect to Δx.
Now, we combine both parts:

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

So, we can have:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

(d)

$$\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \frac{g(x) - g(x+h)}{h} \cdot \frac{1}{g(x+h)g(x)} \longrightarrow -g'(x) \cdot \frac{1}{g(x)^2}$$

where $g(x+h) \to g(x)$ because g is continuous at x. and for sufficiently small h, because $g(x) \neq 0$, and g is continuous at x, so we can have $g(x+h)g(x) \neq 0$.

Prop: (Chain rule) If g is differentiable at x and f is differentiable at $g(x)$, then:

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Pf:

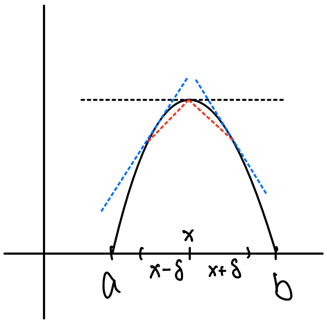
$$\frac{(f \circ g)(x+h) - (f \circ g)(x)}{h} = \begin{cases} \frac{(f\circ g)(x+h)-(f\circ g)(x)}{g(x+h)-g(x)} \cdot \frac{g(x+h)-g(x)}{h}, & g(x+h) \neq g(x) \\ 0 = f'(g(x)) \cdot \frac{g(x+h)-g(x)}{h}, & g(x+h) = g(x) \end{cases}$$

Let $h' := g(x+h) - g(x)$:

$$= \begin{cases} \frac{f(g(x)+h')-f(g(x))}{h'}, & h' \neq 0 \\ f'(g(x)), & h' = 0 \end{cases}$$

$$\cdot \frac{g(x+h) - g(x)}{h} \underset{h \to 0}{\longrightarrow} f'(g(x)) \cdot g'(x)$$

because $h' \to 0$ as $h \to 0$ since g is continuous at x.



Def: Let $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is said to have a **local maximum** at x if $f(x) \geq f(y)$ for all $y \in B_\delta(x) \cap X$ for some $\delta > 0$. Similarly, we define a **local minimum**.

Thm (Fermat): Let $f : [a, b] \to \mathbb{R}$ be a function. If f has a local maximum or minimum and is differentiable at $x \in (a, b)$, then $f'(x) = 0$.

Proof: Suppose f has a local maximum at x, and let $\delta > 0$ be as in the definition above. Then for $x - \delta < x' < x$, we have:

$$\frac{f(x) - f(x')}{x - x'} \geq 0 \quad \text{so, taking } x' \to x, \text{ we get } f'(x) \geq 0.$$

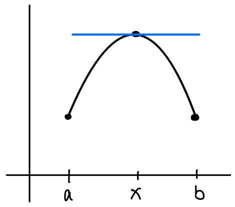
Similarly, for $x < x' < x + \delta$, we have:

$$\frac{f(x) - f(x')}{x - x'} \leq 0 \quad \text{so, taking } x' \to x, \text{ we get } f'(x) \leq 0.$$

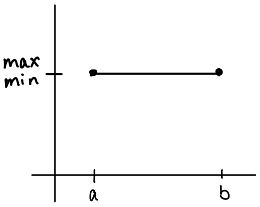
Thus $f'(x) = 0$. The proof for a local minimum is analogous.

Rmk: The converse is not true, e.g., $f(x) = x^3$ at $x = 0$.

Thm (Rolle's Theorem): Let $f \in C([a, b])$. If f is differentiable on (a, b) and $f(a) = f(b)$, then there exists $x \in (a, b)$ such that $f'(x) = 0$.

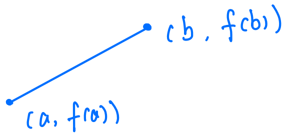


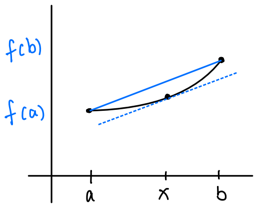
Pf: By the extreme value theorem, f has a (global) maximum and minimum on $[a, b]$. If f has a global extremum at $x \in (a, b)$, then $f'(x) = 0$ by Fermat's Theorem. Otherwise, its global extrema are at the endpoints, which means that f is constant because $f(a) = f(b)$, in which case $f'(x) = 0$ for any $x \in (a, b)$.



Thm (Mean Value Theorem): Let $f \in C([a, b])$. If f is differentiable on (a, b), then there exists $x \in (a, b)$ such that:

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$





Pf: Let $h(t) := f(t) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(t - a)\right]$. Then $h(a) = h(b) = 0$.

By Rolle's Theorem, there exists $x \in (a, b)$ such that $h'(x) = 0$, which implies:

$$f'(x) - \frac{f(b) - f(a)}{b - a} = 0$$

which is what we wanted to show.