Review from Last Class:

Definition

Let $E \subseteq \mathbb{R}$ and suppose $f : E \to \mathbb{R}$ is a function and $p \in E'$. The statement $\lim_{x\to p} f(x) = q$ means that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in (B_\delta(p) \setminus \{p\}) \cap E$, we have $f(x) \in B_\varepsilon(q)$, i.e., $0 < |x - p| < \delta$, $x \in E \implies |f(x) - q| < \varepsilon$.

Proposition

$\lim_{x\to p} f(x) = q$ if and only if for every sequence $(p_n)_{n=1}^{\infty} \subseteq E \setminus \{p\}$ converging to p, we have $f(p_n) \to f(q)$.

Proof:

First, suppose $\lim_{x\to p} f(x) = q$, and let $(p_n)_{n=1}^{\infty} \subseteq E \setminus \{p\}$ be a sequence with $p_n \to p$. Given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - q| < \epsilon$ whenever $0 < |x - p| < \delta$ and $x \in E$.
Furthermore, there exists an $N \geq 1$ such that $0 < |p_n - p| < 1$ whenever $n \geq N$. Thus, if $n \geq N$, we have $0 < |p_n - p| < \delta$, so $|f(p_n) - q| < \epsilon$, which shows that $f(p_n) \to q$.
Converse (Exercise): Hint (Show that if $\lim_{x\to p} f(x) \neq q$, then there exists a sequence $(p_n)_{n=1}^{\infty} \subseteq E \setminus \{p\}$ with $p_n \to p$ such that $f(p_n) \not\to q$).

Proof: Suppose $\lim_{x\to p} f(x) \neq q$. We will show that there exists a sequence $(p_n)_{n=1}^{\infty} \subseteq E \setminus \{p\}$ such that $p_n \to p$ and $f(p_n) \not\to q$.

By the definition of a limit, if $\lim_{x\to p} f(x) = q$, then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in (E \setminus \{p\}) \cap B_\delta(p)$, we have $|f(x) - q| < \varepsilon$.

Since $\lim_{x\to p} f(x) \neq q$, there must exist some $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exists some $x \in (E \setminus \{p\}) \cap B_\delta(p)$ with $|f(x) - q| \geq \varepsilon_0$. For each $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$.

By our assumption, for each n, there exists $p_n \in (E \setminus \{p\}) \cap B_{\frac{1}{n}}(p)$ such that $|f(p_n) - q| \geq \varepsilon_0$.

By construction, $p_n \in B_{\frac{1}{n}}(p)$, which means $|p_n - p| < \frac{1}{n}$. Thus, as $n \to \infty$, we have $p_n \to p$. Since $|f(p_n) - q| \geq \varepsilon_0$ for every n, the sequence $(f(p_n))$ does not converge to q.

If it did, there would exist an $N \in \mathbb{N}$ such that for all $n \geq N$, $|f(p_n) - q| < \frac{\varepsilon_0}{2}$, which contradicts the fact that $|f(p_n) - q| \geq \varepsilon_0 > 0$. Therefore, we have constructed a sequence $(p_n)_{n=1}^{\infty} \subseteq E \setminus \{p\}$ such that $p_n \to p$ and $f(p_n) \not\to q$.

Thus, if $\lim_{x\to p} f(x) \neq q$, there exists a sequence $(p_n)_{n=1}^{\infty} \subseteq E \setminus \{p\}$ with $p_n \to p$ such that $f(p_n) \not\to q$.

Example from Tuesday's Lecture:

$$\lim_{x\to 2} x^2 = 4 \quad (E = \mathbb{R})$$

If $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R} \setminus \{2\}$ and $x_n \to 2$, then by the limit laws, $x_n^2 \to 4$. Limit laws thereby transfer to functions. For example, if $\lim_{x\to p} f(x) = a$ and $\lim_{x\to p} g(x) = b$, then

$$\lim_{x\to p}(f+g)(x) = a + b \quad (f, g : E \to \mathbb{R},\, p \in E').$$

Proof: Let $(p_n) \subseteq E \setminus \{p\}$ with $p_n \to p$. Then $f(p_n) \to a$ and $g(p_n) \to b$. So, by the limit law, $(f+g)(p_n) \to a + b$.

Continuous Functions:

Definition

Let $E \subseteq \mathbb{R}$ and suppose $f : E \to \mathbb{R}$ is a function and $p \in E$ (not E').

Then f is said to be **continuous** at p if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in B_\delta(p) \cap E$, we have:

$$f(x) \in B_\varepsilon(f(p)).$$

Note: Equivalently, if $x \in (B_\delta(p) \setminus \{p\}) \cap E$, then

$$f(x) \in B_\varepsilon(f(p)),$$

because we always have $f(p) \in B_\varepsilon(f(p))$.

If f is continuous at every point of E, we say f is continuous on E.

Observe that if $p \in E \cap E'$, f is continuous at p if and only if: $\lim_{x\to p} f(x) = f(p)$. $p \in E'$ means $B_\delta(p) \setminus \{p\} \cap E \neq \emptyset$ for all $\delta > 0$.

But what if $p \in E \setminus E'$? (These are called **isolated points** of E.)

Note: If $p \notin E'$, it means that there exists a $\delta > 0$ such that $(B_\delta(p) \setminus \{p\}) \cap E = \emptyset$. In other words, no point in E lies arbitrarily close to p except for p itself. If f is continuous at every point of E, we say f is continuous on E.

f is always continuous at p because there exists a $\delta > 0$ such that:

$$B_\delta(p) \cap E = \{p\}$$

and we always have $f(p) \in B_\varepsilon(f(p))$.

Proposition

f is continuous at $p \in E$ if and only if for every sequence $(p_n)_{n=1}^{\infty} \subseteq E$ with $p_n \to p$, we have $f(p_n) \to f(p)$.

Theorem

f is continuous on $\mathbb{R}$ if and only if

$$f^{-1}(F) = \{x \in \mathbb{R} : f(x) \in F\} \text{ is closed for all closed sets } F \subseteq \mathbb{R}.$$

Remark: Equivalently, $f^{-1}(G)$ is open for all open sets $G \subseteq \mathbb{R}$ because $f^{-1}(G)$ is open if $f^{-1}(G^c) = (f^{-1}(G))^c$ is closed. *(Note: check this.)*

Theorem

f is continuous on $\mathbb{R}$ if and only if $f^{-1}(F)$ is closed for all closed sets $F \subseteq \mathbb{R}$.

Proof:
1. **Forward Direction:** Suppose f is continuous on $\mathbb{R}$. We want to show that $f^{-1}(F)$ is closed for all closed sets $F \subseteq \mathbb{R}$.

Let $F \subseteq \mathbb{R}$ be a closed set. To show that $f^{-1}(F)$ is closed, we need to prove that if (x_n) is a sequence in $f^{-1}(F)$ that converges to some point $x \in \mathbb{R}$, then $x \in f^{-1}(F)$.

Suppose $(x_n) \subseteq f^{-1}(F)$ and $x_n \to x$. This means that for each n, $x_n \in f^{-1}(F)$, so $f(x_n) \in F$.

Using the continuity of f: Since f is continuous on $\mathbb{R}$ and $x_n \to x$, we have:

$$f(x_n) \to f(x).$$

Using the closedness of F: Since F is closed and $f(x_n) \in F$ for all n, the limit of the sequence $(f(x_n))$ must also lie in F. Therefore:

$$f(x) \in F.$$

Since $f(x) \in F$, we have $x \in f^{-1}(F)$. Thus, every limit point of $f^{-1}(F)$ is contained in $f^{-1}(F)$, proving that $f^{-1}(F)$ is closed.

2. **Reverse Direction:** Suppose $f^{-1}(F)$ is closed for every closed set $F \subseteq \mathbb{R}$. We want to show that f is continuous on $\mathbb{R}$.

To prove that f is continuous, we need to show that for every $x_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Consider an arbitrary $x_0 \in \mathbb{R}$ and any $\varepsilon > 0$. The set $B_\varepsilon(f(x_0))$, which is the open ball centered at $f(x_0)$ with radius ε, is an open set in $\mathbb{R}$.

The preimage $f^{-1}(B_\varepsilon(f(x_0)))$ is an open set in $\mathbb{R}$ because we assumed that the preimage of every open set under f is open.

Since $x_0 \in f^{-1}(B_\varepsilon(f(x_0)))$, there exists some $\delta > 0$ such that the ball $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$.

This means that for all $x \in B_\delta(x_0)$, we have $f(x) \in B_\varepsilon(f(x_0))$, which is precisely the definition of continuity at x_0.

Since x_0 was arbitrary, f is continuous on $\mathbb{R}$.

Summary of Key Steps:

- The **closedness of** F is used to argue that if $f(x_n) \in F$ for all n and F is closed, then $\lim f(x_n) = f(x)$ must also lie in F.
- The **continuity of** f is used to ensure that $f(x_n) \to f(x)$ whenever $x_n \to x$.

Equivalence of Continuity and Openness/Closeness

A function $f : E \to \mathbb{R}$ is continuous on E if and only if:

- For every closed set $F \subseteq \mathbb{R}$, $f^{-1}(F)$ is closed.
- Equivalently, for every open set $G \subseteq \mathbb{R}$, $f^{-1}(G)$ is open.

Theorem: Compactness of Continuous Functions

Let $K \subseteq \mathbb{R}$ be compact. If $f : K \to \mathbb{R}$ is continuous, then $f(K) = \{f(x) : x \in K\}$ is compact.

Proof

Suppose $\{y_n\} \subseteq f(K)$. For each n, there exists $x_n \in K$ such that $y_n = f(x_n)$. Since K is compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \to x \in K$. Hence, $y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$. Since f is continuous, this shows $f(K)$ is compact.

Remark: We say that $f : E \to \mathbb{R}$ is **bounded** if $f(E)$ is bounded.

Theorem: Extreme Value Theorem

Let $K \subset \mathbb{R}$ be nonempty and compact. If $f : K \to \mathbb{R}$ is continuous, then f is bounded and attains a maximum and minimum on K (i.e., there exist $p, q \in K$ such that

$$f(p) \leq f(x) \leq f(q) \quad \text{for all } x \in K.$$

Proof: From the preceding theorem, we know $f(K)$ is compact. By the Heine-Borel theorem, $f(K)$ is closed and bounded. Since $f(K)$ is nonempty and bounded,

$$m = \inf f(K) \quad \text{and} \quad M = \sup f(K) \quad \text{exist} \quad \text{(Dedekind completeness of } \mathbb{R}\text{)}.$$

Now there exists a sequence $(y_n)_{n=1}^{\infty} \subseteq f(K)$ such that $y_n \to M$ (from MT Q3).
Since $f(K)$ is closed, we must have $M \in f(K)$.
In other words, there exists a $q \in K$ such that

$$f(q) = M = \sup\{f(x) : x \in K\},$$

meaning $f(q)$ is a maximum.
The proof of the existence of a minimum is similar.

Theorems and Properties of Continuous Functions

Theorem: Operations on Continuous Functions

Let f and g be real-valued functions that are continuous at $x_0 \in \mathbb{R}$. Then:

1. $f + g$ is continuous at x_0.
2. fg is continuous at x_0.
3. $\frac{f}{g}$ is continuous at x_0 if $g(x_0) \neq 0$.

Proof:
1. Sum of Continuous Functions:
Let f and g be continuous at x_0. To show $f + g$ is continuous at x_0, take any sequence $(x_n) \to x_0$.
- Since f and g are continuous at x_0:

$$\lim_{n\to\infty} f(x_n) = f(x_0), \quad \lim_{n\to\infty} g(x_n) = g(x_0).$$

- Then:

$$\lim_{n\to\infty} (f(x_n) + g(x_n)) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = f(x_0) + g(x_0) = (f + g)(x_0).$$

Thus, $f + g$ is continuous at x_0.
2. Product of Continuous Functions:
Let f and g be continuous at x_0. Consider the sequence $(x_n) \to x_0$.
- Since f and g are continuous:

$$\lim_{n\to\infty} f(x_n) = f(x_0), \quad \lim_{n\to\infty} g(x_n) = g(x_0).$$

- Then:

$$\lim_{n\to\infty} (f(x_n)g(x_n)) = \lim_{n\to\infty} f(x_n) \cdot \lim_{n\to\infty} g(x_n) = f(x_0) \cdot g(x_0) = (fg)(x_0).$$

Hence, fg is continuous at x_0.
3. Quotient of Continuous Functions:
Assume $g(x_0) \neq 0$ and f and g are continuous at x_0. Consider $(x_n) \to x_0$.
- Then:

$$\lim_{n\to\infty} g(x_n) = g(x_0) \neq 0.$$

- Thus, for sufficiently large n, $g(x_n) \neq 0$ and:

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{\lim_{n\to\infty} f(x_n)}{\lim_{n\to\infty} g(x_n)} = \frac{f(x_0)}{g(x_0)} = \left(\frac{f}{g}\right)(x_0).$$

Therefore, $\frac{f}{g}$ is continuous at x_0.

Theorem: Composition of Continuous Functions

If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.

Explanation of dom(f) and dom(g):
The terms $\text{dom}(f)$ and $\text{dom}(g)$ refer to the domains of the functions f and g, respectively:

$\text{dom}(f)$ is the set of all $x \in \mathbb{R}$ for which the function $f(x)$ is defined.

$\text{dom}(g)$ is the set of all $y \in \mathbb{R}$ for which the function $g(y)$ is defined.
Proof:
We are given that $x_0 \in \text{dom}(f)$ and $f(x_0) \in \text{dom}(g)$. Let (x_n) be a sequence in $\text{dom}(f)$ such that $\lim_{n\to\infty} x_n = x_0$. Since f is continuous at x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. Since g is continuous at $f(x_0)$, we also have $\lim_{n\to\infty} g(f(x_n)) = g(f(x_0))$. Therefore, $g \circ f$ is continuous at x_0.

In the proof, the domains are crucial because, for the composition $g \circ f$ to be defined and continuous at a point $x_0 \in \mathbb{R}$, we require: 1. $x_0 \in \text{dom}(f)$, so $f(x_0)$ is defined. 2. $f(x_0) \in \text{dom}(g)$, so $g(f(x_0))$ is defined.

Examples Illustrating Continuity and Discontinuity

Example 1: Polynomial Function

Let $f(x) = 2x^2 + 1$. Prove that f is continuous on $\mathbb{R}$ using both definitions:

- Using the sequential definition.
- Using the ϵ-δ definition.

Solution:
1. Using the Sequential Definition:
Let (x_n) be a sequence such that $\lim_{n\to\infty} x_n = x_0$.

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(2x_n^2 + 1) = 2\lim_{n\to\infty} x_n^2 + 1 = 2x_0^2 + 1 = f(x_0).$$

Thus, $\lim_{n\to\infty} f(x_n) = f(x_0)$, proving that f is continuous at every $x_0 \in \mathbb{R}$.
2. Using the ϵ-δ Definition:
To show that $f(x) = 2x^2 + 1$ is continuous at any $x_0 \in \mathbb{R}$, we need to find $\delta > 0$ such that for all x with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$.

$$|f(x) - f(x_0)| = |2x^2 + 1 - (2x_0^2 + 1)| = |2x^2 - 2x_0^2| = 2|x - x_0||x + x_0|.$$

Choose $\delta = \min\left(1, \frac{\epsilon}{2(|x_0|+1)}\right)$. Then, if $|x - x_0| < \delta$, we have:

$$|f(x) - f(x_0)| < \epsilon.$$

Explanation of the Choice of δ:
The choice of $\delta = \min(1, \frac{\epsilon}{2(|x_0|+1)})$ ensures that $f(x) = 2x^2 + 1$ is continuous at any $x_0 \in \mathbb{R}$:
1. The term $\min(1, \cdot)$ restricts δ to be at most 1, controlling the size of the neighborhood around x_0.
2. The term $\frac{\epsilon}{2(|x_0|+1)}$ ensures that when $|x - x_0| < \delta$, we have:

$$|f(x) - f(x_0)| = 2|x - x_0||x + x_0| < \epsilon,$$

because $|x + x_0| \le |x - x_0| + 2|x_0| \le 1 + 2|x_0|$ when $|x - x_0| < 1$.
Thus, this choice guarantees both $|x - x_0| < \delta$ and $|f(x) - f(x_0)| < \epsilon$, meeting the criteria for continuity.