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Lecture 10: Compact sets and limits of functions

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10.1 Compact sets

Definition 10.1. A set $K \subseteq \mathbb{R}$ is **(sequentially) compact** if every sequence $(x_n)_{n=1}^{\infty}$ in K has a subsequence $(x_{n_k})_{k=1}^{\infty}$ converging to a point in K.

Example 10.2. The set $[0, 1]$ is compact.

Let $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$. Then $(x_n)_{n=1}^{\infty}$ is bounded, so by the Bolzano–Weierstrass theorem, it has a subsequence $(x_{n_k})_{k=1}^{\infty}$ converging to some $x \in \mathbb{R}$. Moreover, $x \in [0, 1]$ since $[0, 1]$ is closed.

Example 10.3. The set $(0, 1)$ is *not* compact.

Let $(x_n)_{n=1}^{\infty} \subseteq (0, 1)$ be given by $x_n := \frac{1}{n+1}$. If $(x_{n_k})_{k=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$, then $x_{n_k} \to 0$ because $x_n \to 0$, but $0 \notin (0, 1)$.

Example 10.4. The set $[0, \infty)$ is *not* compact.

Let $(x_n)_{n=1}^{\infty} \subseteq [0, \infty)$ be given by $x_n := n$. If $(x_{n_k})_{k=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$, then $x_{n_k} = n_k \geq k$, so $(x_{n_k})_{k=1}^{\infty}$ is not bounded and therefore cannot converge.

Examples 10.3 and 10.4 show that a set may fail to be compact if it is not closed or not bounded. In fact, these two properties are necessary and sufficient for compactness in $\mathbb{R}$.

Theorem 10.5 (Heine–Borel)**.** *Let $K \subseteq \mathbb{R}$. Then K is compact if and only if K is closed and bounded.*

Proof. First, suppose that K is compact. Let $(x_n)_{n=1}^{\infty}$ be a sequence in K converging to some $x \in \mathbb{R}$. Then it has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ converging to some $y \in K$. As all subsequences of a convergent sequence must converge to the limit of the sequence itself, we must have $x = y \in K$, which shows that K is closed. Furthermore, if K were not bounded, there would exist a sequence $(x_n)_{n=1}^{\infty}$ in K with $|x_n| \geq n$. As argued previously, such as sequence cannot have a convergent subsequence since all its subsequences are unbounded. We conclude that K must also be bounded.

Conversely, suppose that K is closed and bounded, and let $(x_n)_{n=1}^{\infty}$ be a sequence in K. By the Bolzano–Weierstrass theorem, it has a convergent subsequence (x_{n_k}) whose limit must be in K since K is closed. □

10.2 Limits of functions

Definition 10.6. Let $E \subseteq \mathbb{R}$. A point $p \in \mathbb{R}$ is called a **limit point** of E if $(B_\delta(p) \setminus \{p\}) \cap E \neq \varnothing$ for all $\delta > 0$. The set of all limit points of E is denoted E'.

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Example 10.7. Let $E \in \mathbb{R} \setminus \{2\}$. Then $2 \in E'$ because for all $\delta > 0$, we have $2 - \frac{\delta}{2} \in B_\delta(2) \setminus \{2\} = (2-\delta, 2) \cup (2, 2+\delta)$ and $2 - \frac{\delta}{2} \in E$.

The following definition is analogous to the definition of $\lim_{n\to\infty} a_n = a$, which means that for all $\varepsilon > 0$, there exists an $N \geq 1$ such that for all $n \geq N$, we have $a_n \in B_\varepsilon(a)$ (i.e., $|a - a_n| < \varepsilon$).

Definition 10.8. Let $E \subseteq \mathbb{R}$ and suppose that $f : E \to \mathbb{R}$ is a function and $p \in E'$. We say that $q \in \mathbb{R}$ is a **limit** of f as x approaches p if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in (B_\delta(p) \setminus \{p\}) \cap E$ (i.e., $0 < |x - p| < \delta$ and $x \in E$), we have $f(x) \in B_\varepsilon(q)$ (i.e., $|f(x) - q| < \varepsilon$). In this case, we write $f(x) \to q$ as $x \to p$, or $\lim_{x\to p} f(x) = q$.

Example 10.9. Let $E = \mathbb{R}$ and $f(x) := x^2$. Then $\lim_{x\to 2} f(x) = 4$.

Let $\varepsilon > 0$. Then $|x^2 - 4| = |x-2||x+2| < \varepsilon$ if $|x - 2| < \frac{\varepsilon}{5}$ and $|x + 2| < 5$. Now if $|x - 2| < 1$, then $|x+2| \leq |x-2| + 4 < 1 + 4 = 5$, so if $\delta := \min\{\frac{\varepsilon}{5}, 1\}$, then $|x^2 - 4| < \varepsilon$ whenever $0 < |x-2| < \delta$ and $x \in \mathbb{R}$.

Example 10.10. Let $E = \mathbb{R} \setminus \{1\}$ and $f(x) := \frac{x^2-1}{x(x-1)}$. Then $\lim_{x\to 1} f(x) = 2$.

Let $\varepsilon > 0$. Then $|\frac{x^2-1}{x(x-1)} - 2| = |\frac{1-x}{x}| = \frac{|1-x|}{|x|} < \varepsilon$ if $|1 - x| < \frac{\varepsilon}{2}$ and $|x| > \frac{1}{2}$. Now if $|x - 1| < \frac{1}{2}$, then $|x| \geq 1 - |1-x| > \frac{1}{2}$, so if $\delta := \min\{\frac{\varepsilon}{2}, \frac{1}{2}\}$, then $|\frac{x^2-1}{x(x-1)} - 2| < \varepsilon$ whenever $0 < |x-1| < \delta$ and $x \in \mathbb{R} \setminus \{1\}$.