POINT SET TOPOLOGY

specifically subset of real numbers.....

Def:⇒ an INTERVAL is a set $I \subseteq \mathbb{R}$ s.t. for all $x, y \in I$, if $x \leq z \leq y$ then $z \in I$.

One can verify that every interval is of one of the following form

open - $(a,b) = \{x \in \mathbb{R} : a < x < b\}$, where $-\infty \leq a \leq b \leq +\infty$

closed - $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, where $-\infty < a \leq b < +\infty$

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Def:⇒ let $x \in \mathbb{R}$ and $r > 0$, the open ball of radius r, centered at x is $B_r(x) = \{y \in \mathbb{R} : |y-x| < r\}$

also denoted $B(x; r)$ $= (x-r, x+r)$

the closed ball of radius r, centered at x is

$\overline{B}_r(x) :\Rightarrow \{y \in \mathbb{R} : |y-x| \leq r\} = [x-r, x+r]$

Rmk⇒ $x_n \xrightarrow{n \to \infty} x$ means for all $\varepsilon > 0$, $\exists$ an $N \geq 1$ s.t. $\forall n \geq N$, $\underbrace{|x - x_n| < \varepsilon}_{x_n \in B_\varepsilon(x)}$

Def :⇒ A set $F \subseteq \mathbb{R}$ is called (SEQUENTAILY) CLOSED, if for all convergent seq in F, the limit is also in F.

in other words⇒ if $(x_n)_{n=1}^{\infty}$ is a seq with $x_n \in F$ an $x_n \to x \in \mathbb{R}$, then $x \in F$

$= (x_n)_{n=1}^{\infty} \subseteq F$

Ex: let $(x_n)_{n=1}^{\infty} \subseteq [a,b]$ where $-\infty < a \le b < +\infty$ and suppose $x_n \to x$.

we know $a \le x_n \le b$ for $\forall n$., so by limit laws $a \le x \le b$

means $x \in [a,b]$

Ex: $(0,1)$ is not closed

let $x_n = \frac{1}{n+1}$, then $(x_n)_{n=1}^{\infty} \subseteq (0,1)$ and $x_n \to 0$

but $0 \notin (0,1)$

which intervals are closed? (a,b) $[a,b]$

$[a,b)$ $(a,b]$

ex of form $(a,b) \Rightarrow (a,a) = \emptyset$ is closed.

$(-\infty, +\infty) = \mathbb{R}$ is closed

fundamental prop. of closed sets :

a) Arbitrary intersections of closed sets are closed

b) finite unions of closed sets are closed.

pf ⇒ a) let F_i be a closed set for each $i \in I$ where I is some index set

we will show that $\bigcap_{i \in I} F_i$ is closed.

Consider $(x_n) \subseteq \bigcap_{i \in I} F_i$ and $x_n \to x$

for each $i \in I$, we have $x_n \subseteq F_i$, so $x \in F_i$ b/c F_i is closed

hence $x \in \bigcap_{i \in I} F_i$

b) let $F_1, F_2, \dots, F_M$ be closed sets

~~suppose $x_n \subseteq \bigcup F_i$~~

Suppose $x_n \subseteq \bigcup_{i=1}^{M} F_i$ with $x_n \to x$

One of these sets must contain infinitely many terms of this seq (x_n)

Say F_1, so we have a subseq. $(x_{n_k})_{k=1}^{\infty} \subseteq F_1$

since $x_{n_k} \to x$ and F_1 is closed

we must have $x \in F_1 \subseteq \bigcup_{i=1}^{M} F_i$

Ex: is $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1-\frac{1}{n}\right]$ closed?

Ans ⇒ not closed

consider $x_n = \frac{1}{n}$ ⇒ then $x_n \in \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1-\frac{1}{n}\right]$ for each n.

since for $n \to \infty$ ⇒ we have interval ⇒ $(0, 1)$

approaching these values but never reach

~~However when n=2,3,4~~

claim $0 \in \left[\frac{1}{n}, 1-\frac{1}{n}\right]^n = \{x \in \mathbb{R} ; \frac{1}{n} \leq x \leq 1-\frac{1}{n}\}$

$0 < \frac{1}{n}$ always true.

hence $0 \notin \left[\frac{1}{n}, 1-\frac{1}{n}\right]$

$\therefore 0 \notin \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1-\frac{1}{n}\right]$

however $\frac{1}{n} \to 0$ ∴ limit not in interval.

Def: a set $E \subseteq \mathbb{R}$ is <u>open</u> if $E^c = (\mathbb{R} \setminus E)$ is closed.

Ex: if $-\infty < a \leq b < +\infty$ then $(a,b)^c = \underbrace{\underbrace{(-\infty, a]}_{\text{closed}} \cup \underbrace{[b, +\infty)}_{\text{closed}}}_{\text{closed}}$

so (a,b) is open.

for $(-\infty, b)^c = \underbrace{[b, \infty)}_{\text{closed}}$ $\therefore$ $(-\infty, b)$ is open.

$(a, +\infty)^c = \underbrace{(-\infty, a]}_{\text{closed}}$ $\therefore$ $(a, +\infty)$ is open.

Rmk: $\mathbb{R} = (-\infty, +\infty)$ is open and closed

$\emptyset$ is open and closed

$(0,1]$ is not closed and $(0,1]^c = (-\infty, 0] \cup (1, \infty)$ is not closed

so $(0,1]$ is neither.

Prop of OPEN SETS:

a) Arbitrary unions of open sets are open

b) Finite intersections of open sets are open.

Pf:⇒ a) let E_i be an open set for each $i \in I$. then $\left(\bigcup_{i \in I} E_i\right)^c$ (de morgans law) $= \bigcap_{i \in I} E_i^c$

since E_i is open
by def. E_i^c is closed ∴

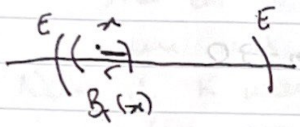
$\bigcap_{i \in I} E_i^c$ is closed, hence $\bigcup_{i \in I} E_i$ is open.

b) Similar

Rmk:⇒ $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$ is not open!

Prop:⇒ let $E \subseteq \mathbb{R}$. E is open iff

for $\forall x \in E$, $\exists r>0$, s.t. $B_r(x) \subseteq E$



Pf⇒ first suppose E is open
means $F = E^c$ is closed and let $x \in E$

for the sake of contr., suppose
for $\forall r>0$ that $B_r(x) \not\subseteq E$
then for each $n \geq 1$, $\exists$ an $x_n \in B_{\frac{1}{n}}(x)$ s.t. $x_n \in E^c = F$

(here we take $r = \frac{1}{n}$), Since $|x_n - x| < \frac{1}{n}$ we have $x_n \to x$

but F is closed, so $x \in F$ however contradicts
initial assumption that $x \in E$ ⇒ because $E^c = F$ ∴
$E \neq F$.