LEC8 8.20

SERIES

Def Let $(a_n)_{n=1}^{\infty}$ be a seq. in $\mathbb{R}$. The SERIES $\sum_{n=1}^{\infty} a_n$ (or $\sum_n a_n$, $\sum a_n$) is said to CONVERGE if the seq $\left(\sum_{n=1}^{N} a_n\right)_{N=1}^{\infty} = (a_1, a_1+a_2, a_1+a_2+a_3, \dots)$ of PARTIAL SUMS converges.

Otherwise, it is said to be DIVERGE.

If $\left(\sum_{n=1}^{N} |a_n|\right)_{N=1}^{\infty}$ conv, the series is said to CONVERGE ABSOLUTELY.

Rmk: Let $S_N := \sum_{n=1}^{N} a_n$. Since (S_N) is conv. iff it is Cauchy, we see that $\sum_{n=1}^{\infty} a_n$ conv. iff

[for all $\varepsilon > 0$, there exists an $N \geq 1$ s.t. for all $n, m \geq N$. We have $|S_n - S_m| < \varepsilon$.]

WLOG $n \geq m$. $|S_n - S_m| = \left|\sum_{k=m+1}^{n} a_k\right|$

Rmk: As a result, abs. conv. implies conv. b/c $\left|\sum_{k=m+1}^{n} |a_k|\right| = \sum_{k=m+1}^{n} |a_k| < \varepsilon$, then

$$\left|\sum_{k=m+1}^{n} a_k\right| \underset{\Delta \text{ ineq.}}{\leq} \sum_{k=m+1}^{n} |a_k| < \varepsilon$$

$(a_n)_{n=1}^{\infty}$ $(a_1, a_2, \dots)$ iff converge

$\Updownarrow$ $\Updownarrow$

$(a_{m+1})_{n=1}^{\infty}$ $(a_1, a_2, \dots)$

Prop. If $\sum a_n$ conv., then $a_n \to 0$

Pf: In the Cauchy criterion, take $n = m+1$ to get that for all $\varepsilon > 0$, there exists an $N \geq 1$ s.t. for all $m \geq N$, we have $|a_{m+1}| < \varepsilon$, which shows that $a_{m+1} \to 0$, or equivalently $a_n \to 0$

This is called the DIVERGENCE TEST in its contrapositive form: if $a_n \not\to 0$, then $\sum a_n$ diverges.

Ex. (Geometric Series) Let $r \in \mathbb{R}$. Then the series $\sum_{n=0}^{\infty} r^n$ cvg? dvg?

Sol. If $|r| \geq 1$, then $|r^n| = |r|^n \geq 1$ for all n, so $r^n \not\to 0$, Hence the series diverges.

If $|r| < 1$, then $\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r} \xrightarrow[N\to\infty]{\text{HW}} \frac{1}{1-r}$ $\quad (r^0 + r^1 + \dots + r^N) - r(r^0 + r^1 + \dots + r^N) = 1 + r^{N+1}$

Hence the series converges.

In this case, we write $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

Thm (Comparison Test) Suppose $|a_n| \leq c|b_n|$ for some $c > 0$ and all $n \geq N$.

If $\sum |b_n|$ conv, then $\sum |a_n|$ conv (equivalently, if $\sum |a_n|$ div. then $\sum |b_n|$ div)

Pf: Given an $\varepsilon > 0$. there exists an $N' \geq 1$ s.t. for all $n \geq m \geq N'$, we have $\sum_{k=m+1}^{n} |b_k| < \frac{\varepsilon}{C}$

Hence for $n \geq m \geq \max\{N, N'\}$, we have $\sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} C|b_k| < \varepsilon$

This shows that the Cauchy criterion for the conv. of $\sum |a_n|$ is satisfied.

Thm (Cauchy condensation test) Suppose that (a_n) is a decreasing seq. of nonnegative real numbers. Then $\sum_{n=1}^{\infty} a_n$ conv. iff $\sum_{k=0}^{\infty} 2^k \cdot a_{2^k} = a_1 + 2a_2 + 4a_4 + \dots$ conv.

Pf: Let $S_N := \sum_{n=1}^{N} a_n$ and $T_K := \sum_{k=0}^{K} 2^k \cdot a_{2^k}$

Note that (S_N) and (T_K) are both increasing since $a_n \geq 0$

Now given an $N \geq 1$, choose a $K \geq 0$ s.t. $N < 2^{K+1}$. Then $S_N \leq a_1 + a_2 + a_3 + a_4 + \dots + a_7 + \dots + a_{2^K} + \dots + a_{2^{K+1}-1}$

$$S_N \leq \underbrace{a_1} + \underbrace{a_2 + a_3} + \underbrace{a_4 + a_5 + a_6 + a_7} + \dots + a_{2^K} + \dots + a_{2^{K+1}-1}$$

$$\leq a_1 + 2a_2 + 4a_4 + \dots + 2^K a_{2^K} \quad \text{b/c } (a_n) \text{ is decreasing}$$

$$= T_K$$

Thus, if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ conv., the seq of partial sums (T_K) conv., and is therefore bdd.

Hence (S_N) is also bdd and therefore converges by the MCT. meaning that $\sum_{n=1}^{\infty} a_n$ conv.

Similarly, one can show that $T_K \leq 2S_N$ for $2^K \leq N$ (check!)

Ex (p-series) Let $p \in \mathbb{Z}$ (more generally, $p \in \mathbb{Q}$ or $p \in \mathbb{R}$) Then $\sum_{n=1}^{\infty} \frac{1}{n^p} \dots$

Hint: Apply Cauchy condensation test

① $p \geq 0$ $\quad \sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} (2^{1-p})^k$ $\quad$ conv. if $2^{1-p} < 1 \iff p > 1$

div. if $2^{1-p} \geq 1 \iff p \leq 1$

② $p < 0$ div.

Rmk. converges iff $p > 1$

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ (HARMONIC SERIES) diverges — despite the fact that $\frac{1}{n} \to 0$

$(p=1)$

Thm (Root test)

Let (a_n) be a seq. in $\mathbb{R}$ and $\rho := \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$

(a) If $\rho < 1$, then $\sum |a_n|$ conv.

(b) If $\rho > 1$, then $\sum a_n$ div

but this series dvg

Rmk. If $\rho = 1$, the test is inconclusive: for $\sum \frac{1}{n}$ we have $\left(\frac{1}{n}\right)^{\frac{1}{n}} = \frac{1}{n^{\frac{1}{n}}} \to 1 =: \rho$

for $\sum \frac{1}{n^2}$ we have $\left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \frac{1}{n^{\frac{1}{n}}} \cdot \frac{1}{n^{\frac{1}{n}}} \to 1$

this series cvg ($p=2$)

Pf (a) If $\rho < 1$. Let r be s.t. $0 \le \rho < r < 1$. Then by property of lim sup.

there exists an $N \geq 1$ s.t. for all $n \geq N$. we have $|a_n|^{\frac{1}{n}} < r$ i.e., $|a_n| < r^n$ $(n \geq N)$

Hence $\sum |a_n|$ conv. by comparison to $\sum r^n$, which conv.

(b) We prove the contrapositive. If $\sum a_n$ conv., then $a_n \to 0$

So there exists an $N \geq 1$ s.t. for all $n \geq N$. we have $|a_n| < 1$ (any pos # works here)

Hence $|a_n|^{\frac{1}{n}} < 1$ for $n \geq N$. which implies that $\sup_{m \geq n} |a_m|^{\frac{1}{m}} \leq 1$ for $n \geq N$. Therefore,

$$\rho = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \sup_{m\geq n} |a_m|^{\frac{1}{m}} \leq 1$$

Ex (Power Series) A POWER SERIES centred at $x_0 \in \mathbb{R}$ is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n, \text{ where } c_n \in \mathbb{R} \text{ are constants and } x \text{ is a variable}$$

The conv. of this series can be determined using the root test