lec notes 8/19/21

Thm (Dedekind completeness / least upper bound prop of $\mathbb{R}$) Let $A \subseteq \mathbb{R}$ be non-$\emptyset$ and bd. above. Then $\sup(A)$ exists. (Least upper bound)
Similarly, if $A \subseteq \mathbb{R}$ is non-$\emptyset$ & bd. below, then $\inf(A)$ exists. (Greatest lower bound)

Remark
We say $S \subseteq \mathbb{R}$ is **bounded** if $\exists r \in \mathbb{R}_{>0}\ \forall s \in S\ (|s| < r)$

S is bd. iff it is bd. above and below.
$-r < s < r$ $\quad S \le u$ $\quad S \ge \ell$ $\quad s \le u \wedge s \ge \ell \Rightarrow |s| < \max\{|u|, |\ell|\} + 1$

Def A sequence (x_n) in $\mathbb{R}$ is **increasing** if $x_n \le x_{n+1}\ \forall n$ and **decreasing** if $x_n \ge x_{n+1}\ \forall n$.

Remark Sometimes "nondecreasing" is used to mean $x_n \le x_{n+1}$ and "increasing" to mean $x_n < x_{n+1}$. We will call these increasing and strictly increasing, respectively.

Remark Increasing/decreasing seq. are called **monotone/monotonic**

Thm (Monotone seq. thm.) If (x_n) is increasing & bd. above, then $\lim_{n\to\infty} x_n = \sup_{n\ge 1} x_n$ $(= \sup\{x_n : n \ge 1\})$

Pf: Exercise (Hint): Let $x := \sup_{n\ge 1} x_n$. Given $\varepsilon > 0$, the num $x - \varepsilon$ is not an UB of $\{x_n : n \ge 1\}$.

Subsequences:

Def Let $(x_n)_{n=1}^{\infty}$ be a seq. A subsequence of $(x_n)_{n=1}^{\infty}$ is a seq. of the following form $\xrightarrow{k=1}$ $(x_{n_k})_{k=1}^{\infty}$ where $(n_k)_{k=1}^{\infty}$ is a strictly increasing seq in $\mathbb{Z}_{>0}$

Ex: $(x_1, x_3, x_5, \ldots)$ and $(x_2, x_4, x_6, \ldots)$
$n_1 < n_2 < n_3 < \cdots$ $\quad$ $n_1 < n_2 < n_3 < \cdots$

$(x_1, x_3, x_2, x_4, \ldots)$ is NOT a sub. seq. OR $(x_1, x_3, x_3, x_4, \ldots)$
$3 \not< 2$, $n_2 \not< n_3$ $\quad$ $n_2 \not< n_2$

Rmk $n_k \geq k$ in a subseq. ($n_1 \geq 1$ and $n_k \geq k \Rightarrow n_{k+1} \geq n_k + 1 \geq k+1$)

Prop Let (x_n) be a seq, then $x_n \to x$ iff every subseq of (x_n) conv. to x

Pf: (1) $\nexists$ (2) shown correct in class. These are 2 proofs for the above proposition.

Pf:

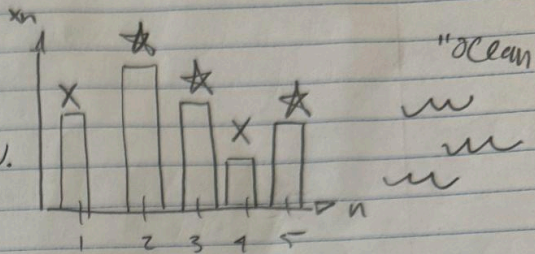
(1) Suppose $(x_{n_k})_{k=1}^{\infty}$ is a subseq. of (x_n), WTS: $x_n \to x$ iff all subseq. of x_n conv. to x
Let $\varepsilon > 0$, WTS $|x_{n_k} - x| < \varepsilon$. Then $\exists N \in \mathbb{N}_{>0}$, $\forall n \geq N$ $(|x_n - x| < \varepsilon)$
$\forall k \geq N$, $(n_k \geq N)$ because $n_k \geq k$.
So, $|x_{n_k} - x| < \varepsilon$, thus, $\lim_{k \to \infty} x_{n_k} \to x$ $\forall$ subseq, thus $x_n \to x$. ■

(2) Pf: Suppose that every subsequence of x_n conv. to x. WTS: $x_n \to x$ iff
Let $(x_n)_{n=1}^{\infty}$ be a seq and subseq $(x_1, x_2, x_3, \ldots)$
Since $\forall$ subseq of x_n is conv. to x, and
$(x_1, x_2, x_3, \ldots) = (x_n)_{n=1}^{\infty}$, x_n is conv. to x. ■

Lemma Let (x_n) be a seq. Then (x_n) has a monotone subseq.

Pf: "Spanish hotel argument"

Suppose hotel n has height x_n.
Hotel m blocks view of hotel n if $n<m$ and $x_n \leq x_m$.
A hotel gets a ★ for unblocked view.
If there are inf. many ★'s, say $n_1 < n_2 < n_3 < \dots$, we pick these indices for our subsequence.

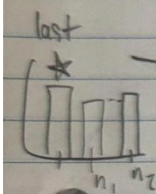


Case 1

Then $x_{n_1} > x_{n_2} > x_{n_3} > \dots$

Case 2

On the other hand, if there are finitely many ★'s, eventually they must stop beyond a certain point. Let n_1 be the hotel after the last ★. Since it has no ★, there must be an $n_2 > n_1$ w/ $x_{n_1} \leq x_{n_2}$. Similarly, there must be an $n_3 > n_2$ w/ $x_{n_2} \leq x_{n_3}$, etc, which is increasing. (Formal Proof)



$n_2 > n_1$
$x_{n_2} \geq x_{n_1}$

Pf: (want to show)

Assuming (x_n) is a seq., WTS $\exists$ monotone subseq. (incr. or dec.)
Define subseq (x_{n_k}) s.t. $n_1 = 1$, for each $k \geq 1$, if $\exists n_{k+1} > n_k$ s.t. $x_{n_{k+1}} > x_{n_k}$, set n_{k+1} to smallest index. Otherwise, let n_{k+1} be smallest index s.t. $x_{n_{k+1}} \leq x_{n_k}$.

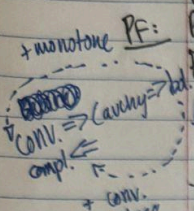
Case 1:
If the indexing above continues infinitely w/ $x_{n_{k+1}} > x_{n_k}$ $\forall k$, then (x_{n_k}) is a strictly incr. subseq. of (x_n)

Case 2:
If eventually no n_{k+1} can be found where $x_{n_{k+1}} > x_{n_k}$, from this point on, $(x_{n_k} \geq x_{n_{k+1}} \geq \dots \; \forall k)$ then (x_{n_k}) is a decreasing subseq. of (x_n).

In both cases, $\exists$ a monotone subseq within (x_n), incr. or decr, thus every seq. (x_n) has a monotone subseq.

∎

Thm | (Bolzano-Weierstrass)
If (x_n) is bd. it has a conv. seq.
Pf: By the Lemma, it has a monotone subseq.
By the MST, this subseq. converges.
Ex: $x_n = (-1)^n$ $-1, +1, -1, +1, \ldots$
All even/odd are conv. subseq.



MT

Extended Real Numbers:

Def |
The extended real numbers are $\mathbb{R} \cup \{-\infty, +\infty\}$. We define $-\infty \leq x \leq +\infty$ $\forall x \in \mathbb{R} \cup \{\pm\infty\}$
Rmk | We will not def alg. ops. w/ $\pm\infty$.

Previously we saw if $A \subseteq \mathbb{R}$ is non-$\emptyset$ and bd. above, then $\sup(A)$ exists.
In fact, if $A \subseteq \mathbb{R} \cup \{\pm\infty\}$, then $\sup(A)$ always exists. (as will $\inf(A)$) — verify! ✱
Ex: $\sup \emptyset = -\infty$, $\sup\{1, 2, 3, \ldots\} = +\infty$

Def | let (x_n) be a seq. in $\mathbb{R} \cup \{\pm\infty\}$. We write $x_n \to +\infty$ if
$\forall a \in \mathbb{R}$ $\exists N \in \mathbb{Z}_{>0}$ $\forall n \geq N$ $(x_n > a)$ (x_n is larger than a)
$x_n \to -\infty$ if $\forall a \in \mathbb{R}$ $\exists N \in \mathbb{Z}_{>0}$ $\forall n \geq N$ $(x_n < a)$
Warning: the limit laws we prev. prooved don't necessarily apply!
Ex: $\underbrace{n^2}_{\to +\infty} - \underbrace{n}_{\to +\infty} \longrightarrow +\infty$, not $(+\infty - (+\infty))$

Ex: $n^2 \to +\infty$

Pf: Let $a \in \mathbb{R}$. Then $x_n = n^2 \geq n > a$ if $n > a$. By the A.P., $\exists N \in \mathbb{Z}_{>0}$ s.t. $N > a$. Hence, if $n \geq N$, then $x_n > a$.

Ex: $\frac{n^2+3}{n+1} \to +\infty$

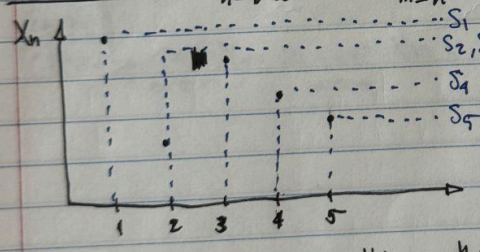
Pf: Let $a \in \mathbb{R}$, then $x_n = \frac{n^2+3}{n+1} \geq \frac{n^2}{n+1} \geq \frac{n^2-1}{n+1} \geq n-1 > a$ if $n > a+1$. By the A.P., $\exists N \in \mathbb{Z}_{>0}$ s.t. $N > a+1$. Hence if $n \geq N$, then $x_n > a$.

Rmk: With this def, MST and BWT must hold w/o assumption of boundedness b/c an unbound monotone seq. conv. to $+\infty$ or $-\infty$.

Def: Let (x_n) be a seq in $\mathbb{R} \cup \{\pm\infty\}$. The limit superior of (x_n) is denoted $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{m \geq n} x_m) = \lim_{n\to\infty}$

If $s_n := \sup_{m \geq n} x_m$, then $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} s_n$

Limit inferior: $\liminf_{n\to\infty} x_n := (\inf_{m \geq n} x_m) = \lim_{n\to\infty}$



Ex: $(x_n) = (1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots)$

$s_1 = 1$

$s_2, s_3 = 1/3$

$s_4, s_5 = 1/5$

Notice that $s_n \geq s_{n+1}$ b/c $\{x_m : m \geq n\} \supseteq \{x_m : m \geq n+1\}$

$\Rightarrow \sup_{m \geq n} x_m \geq \sup_{m \geq n+1} x_m$

Therefore lim sup and lim inf always exist in $\mathbb{R} \cup \{\pm\infty\}$

bounded seq. in $\mathbb{R}$, lim sup and lim inf always exist in $\mathbb{R}$

Hence, by the MST, $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} s_n = \inf_{n \geq 1} s_n = \inf_{n \geq 1} \sup_{m \geq n} x_m$

Similarly, $\liminf_{n\to\infty} x_n = \sup_{n \geq 1} \inf_{m \geq n} x_m$

Ex: $x_n = -1, +1, -1, +1, \ldots$

$S_1 = 1, \; S_2 = 1, \ldots$

$\limsup_{n\to\infty} x_n = +1 \qquad \liminf_{n\to\infty} x_n = -1$

Prop: Let (x_n) be a seq in $\mathbb{R} \cup \{\pm\infty\}$

If $\limsup_{n\to\infty} x_n = s \in \mathbb{R}$, then $\forall \varepsilon > 0, \exists N \geq 1, \forall n \geq N$

$(x_n < \underbrace{s+\varepsilon}_{\text{slightly bigger than lim sup}})$

Pf:

Let S_n be defined as before, given an $\varepsilon > 0$, $\exists N \geq 1$ s.t. $|S - S_n| < \varepsilon \; \forall n \geq N$ b/c $S_n \to S$ by def[redacted] of limsups.

Hence for $m \geq N$, we have $x_m \leq \sup_{m \geq N} x_m = S_n < s + \varepsilon$

Prop: Let (x_n) be a seq in $\mathbb{R} \cup \{\pm\infty\}$ and suppose $\ell \in \mathbb{R}$ (we could also take $\ell \in \mathbb{R} \cup \{\pm\infty\}$).

Then $\lim x_n = \ell \iff \liminf x_n = \limsup x_n = \ell$

Rmk:

Clearly $\inf_{m \geq n} x_m \leq \sup_{m \geq n} x_n$, so $\liminf x_n \leq \limsup x_n$

monotone seq. Thm

If (x_n) is incr. and bd. above, then $\lim_{n\to\infty} x_n = \sup_{n\geq 1} x_n$

$(= \sup\{x_n : n \geq 1\})$

Pf:

Let $x = \sup\{x_n : n \geq 1\}$. By def of supremum, x is the UB of $\{x_n : n \geq 1\}$, meaning $x_n \leq x \ \forall n \geq 1$ and $\forall \varepsilon > 0$, $x - \varepsilon$ is not an UB of $\{x_n : n \geq 1\}$.

Let all $\varepsilon > 0$.

Since $x - \varepsilon$ is not an UB, $\exists N \geq 1$ s.t. $x_N > x - \varepsilon$.

Because (x_n) is increasing ($x_n \leq x_{n+1} \ \forall n$), we have $x_n \geq x_N > x - \varepsilon \ \forall n \geq N$. This can be expressed as $x - \varepsilon < x_n \leq x$.

The above ineq. shows $\forall n \geq N$, $|x_n - x| < \varepsilon$, implying x_n is eventually within ε of x. Since $\varepsilon > 0$, we can say $\lim_{n\to\infty} x_n = x$.

Therefore, $\lim_{n\to\infty} x_n = \sup\{x_n : n \geq 1\}$ ▨

If $A \subseteq \mathbb{R} \cup \{\pm\infty\}$, then sup(A) always exists (as well as inf(A))

Verification:

Pf:

Let A be non-$\emptyset$ subset of $\overline{\mathbb{R}}$ ($\mathbb{R} \cup \{\pm\infty\}$).

Case 1:

If $+\infty \in A$, then $+\infty$ is an UB of A, since nothing is greater in extended real nums. Therefore $\sup(A) = +\infty$

Case 2:

If $+\infty \notin A$, and A is bounded above by some $M \in \mathbb{R}$, then $A \subseteq \mathbb{R}$, and by the LUB prop of $\mathbb{R}$, sup(A) exists and is a real num; if A was not bd. above, it implies $\forall M \in \mathbb{R}$, $\exists a \in A$ s.t. $a > M$, leading to $\sup(A) = +\infty$, even if $+\infty$ isn't in A.

*And we are assuming $+\infty$ isn't in A

Similarly, to prove inf(A) exists if $A \subseteq \mathbb{R} \cup \{\pm\infty\}$, let A be a non-$\emptyset$ subset of $\mathbb{R} \cup \{\pm\infty\}$

Case 1: If $-\infty \in A$, then $-\infty$ is a LB of A, since nothing is less than $-\infty$ in $\mathbb{R} \cup \{\pm\infty\}$. Therefore $\inf(A) = -\infty$

Case 2:

If A is bd. below by some $m \in \mathbb{R}$, then $A \subseteq \mathbb{R}$. By the GLB prop of $\mathbb{R}$, inf(A) exists and is a real number. If A is not bd. below and $-\infty$ is not in A, it implies $\forall m \in \mathbb{R}$, $\exists a \in A$ s.t $a < m$. This shows $\inf(A) = -\infty$ even when $-\infty$ isn't in A.

Thus, we have proven if $A \subseteq \mathbb{R} \cup \{\pm\infty\}$.

■