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Lecture 4: Sequences

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4.1 Triangle Inequality

Proposition 4.1 (Triangle Inequality)**.** *If* $a, b \in \mathbb{Q}$, *then* $|a + b| \leq |a| + |b|$.

Proof. Clearly, $a \leq |a|$ and $b \leq |b|$. Hence, $a + b \leq |a| + |b|$. Similarly, $-a \leq |a|$ and $-b \leq |b|$, so $-(a + b) \leq |a| + |b|$. Thus, $|a + b| = \max(a + b, -(a + b)) \leq |a| + |b|$. □

Proposition 4.2 (Reverse Triangle Inequality)**.** *If* $a, b \in \mathbb{Q}$, *then* $||a| - |b|| \leq |a - b|$.

Proof. By the triangle inequality, $|(a+b)-b| \leq |a+b|+|b|$, so $|a|-|b| \leq |a+b|$. Symmetrically, $|b|-|a| \leq |a+b|$. Hence, $||a| - |b|| = \max(|a| - |b|, |b| - |a|) \leq |a| + |b|$. □

4.2 Definition of Convergence

Definition 4.3. A sequence $(q_n)_{n=1}^{\infty}$ in $\mathbb{Q}$ **converges** to $q \in \mathbb{Q}$ if for all $\varepsilon > 0$ there exists a natural number $N > 0$ such that for all $n \geq N$, $|q - q_n| < \varepsilon$. We call q the **limit** of (q_n) and write $q_n \to q$.

Example 4.4. $q_n \coloneqq \frac{1}{n}$. Then $q_n \to 0$.

Proof. Let $\varepsilon \in \mathbb{Q}_{>0}$. Then $|0 - q_n| < \varepsilon$ whenever $n > \frac{1}{\varepsilon}$. Rewrite $\frac{1}{\varepsilon}$ as $\frac{a}{b}$. If $N = a + 1$, then $N > a \geq \frac{a}{b} = \frac{1}{\varepsilon}$. Hence, if $n \geq N$, $|0 - q_n| < \varepsilon$. □

4.3 Archimedean Property in Convergence Proofs

Proposition 4.5 (Archimedean Property)**.** *For all* $q \in \mathbb{Q}$, *there exists a natural number* $N \in \mathbb{N}$ *with* $N > q$.

Proof. We will break this into two cases. First, if $q \leq 0$, then $N = 1 > q$. Otherwise, if $q > 0$, we may write $q = \frac{a}{b}$ for positive $a, b \in \mathbb{N}$. Then, if $N = a + 1$, $N > a \geq \frac{a}{b} = q$. □

Example 4.6. Let $q_n \coloneqq 1 - \frac{2}{3n}$. Then $q_n \to 1$.

Proof. Let $\varepsilon > 0$. We want $|1 - (1 - \frac{2}{3n})| = |\frac{2}{3n}| < \varepsilon$.
This surely occurs whenever $n > \frac{2}{3\varepsilon}$. By the Archimedean Property, there exists an $N \in \mathbb{N}$ such that $N > \frac{2}{3\varepsilon}$. Then $n \geq N$ would yield $n > \frac{2}{3\varepsilon}$, completing our proof. □

Example 4.7. Let $q_n \coloneqq \frac{3n^2}{n^2+1}$. Then $q_n \to 3$.

Proof. Let $\varepsilon \in \mathbb{Q}_{>0}$. We desire $|3 - \frac{3n^2}{n^2+1}| = |\frac{3}{n^2+1}| < |\frac{3}{n}| < \varepsilon$.
By the Archimedean Property, we obtain an $N \in \mathbb{N}$ such that $N > \frac{3}{\varepsilon}$. Thus, $n \geq N$ yields $|3 - q_n| < \varepsilon$. □

Example 4.8. Let $q_n \coloneqq \frac{2n-1}{3n+2}$. Then $q_n \to \frac{2}{3}$.

Proof. Let $\varepsilon \in \mathbb{Q}_{>0}$. Then we want $|\frac{2}{3} - q_n| = |\frac{2}{3} - \frac{2n-1}{3n+2}| = |\frac{7}{9n+6}| < |\frac{7}{9n}| < \varepsilon$.
By the Archimedean Property, we select a positive integer $N > \frac{7}{9\varepsilon}$. Hence, $n \geq N$ yields $|\frac{7}{9n}| < \varepsilon$, and, in turn, $|\frac{2}{3} - q_n| < \varepsilon$, completing our proof. □

Proposition 4.9 (Uniqueness of limits)**.** *Let $(q_n)_{n=1}^{\infty}$ be a convergent sequence. If both $q_n \to q$ and $q_n \to q'$, then $q = q'$.*

Proof. Towards a contradiction, suppose $q \neq q'$. Let $\varepsilon = \frac{|q-q'|}{2}$. Since $q_n \to q$, there exists an $N_1 \in \mathbb{N}$ such that $|q - q_n| < \varepsilon$ for all $n \geq N_1$. Similarly, there exists an $N_2 \in \mathbb{N}$ such that $|q' - q_n| < \varepsilon$ for all $n \geq N_2$. Hence, for all $n \geq \max(N_1, N_2)$, we obtain $|q - q'| \leq |q - q_n| + |q' - q_n| < 2\varepsilon = |q - q'|$. Since we have arrived at a contradiction, it must be that $q = q'$. □

4.4 Divergent Sequences

Example 4.10. $q_n \coloneqq (-1)^n$ does *not* converge.

Proof. For contradiction, suppose $q_n \to q$. Let $\varepsilon = 1$. By the definition of convergence, there must exist an $N \in \mathbb{N}$ such that $|q - q_n| < 1$ for all $n \geq N$. Thus, $|q_{2N} - q| + |q_{2N+1} - q| < 2$. By the triangle inequality, it follows that $2 = |(-1) - 1| = |q_{2N+1} - q_{2N}| < 2$, a contradiction. □

Example 4.11. $q_n \coloneqq n^2$ does *not* converge.

Proof. For contradiction, suppose $q_n \to q$. Let $\varepsilon = 1$. By the definition of convergence, for some $N \in \mathbb{N}$, $|q - q_n| < \varepsilon$ for all $n \geq N$. However, this implies that $|q_{N+1} - q_N| \leq |q - q_{N+1}| + |q - q_N| < 2\varepsilon = 2$. Then $2N + 1 = |(N^2 + 2N + 1) - N^2| < 2$, which is impossible since $N \geq 1$. Hence, (q_n) must diverge. □

Remark. In the previous example, one can understand that (q_n) does not converge since its elements grow arbitrarily large. That is, the sequence (q_n) is **unbounded**.

Definition 4.12. A sequence $(q_n)_{n=1}^{\infty}$ is **bounded** if, for some $r \in \mathbb{Q}_{>0}$, for all $n \geq 1$, $|q_n| < r$.

Remark. Negating the above definition, we find that a sequence $(q_n)_{n=1}^{\infty}$ is **unbounded** if for all $r \in \mathbb{Q}_{>0}$, there exists an $n \geq 1$ such that $|q_n| \geq r$.

Proposition 4.13. *Let $(q_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{Q}$. If $q_n \to q$, then (q_n) is bounded.*

Proof. Let $\varepsilon = 1$. Since $q_n \to q$, there exists a natural number N such that $|q - q_n| < 1$ for all $n \geq N$. Hence, whenever $n \geq N$, $|q_n| \leq |q_n - q| + |q| < |q| + 1$. If $n < N$, $|q_n| < |q_n| + 1$.
Define $r = \max\{|q_1| + 1, |q_2| + 1, \ldots, |q_{N-1}| + 1, |q| + 1\}$. It directly follows that $|q_n| < r$ for all $n \in \mathbb{N}$. □

Corollary 4.14. *If $(q_n)_{n=1}^{\infty}$ is unbounded then it does not converge.*

Proof. This is the contrapositive of Proposition 4.13. □

Example 4.15. As in Example 4.11, let $q_n := n^2$. Then $(q_n)_{n=1}^{\infty}$ does not converge.

Proof. By Corollary 4.14, it suffices to show that (q_n) is unbounded. Consider an arbitrary $r \in \mathbb{Q}_{>0}$. By the Archimedean Property, there exists a natural number $N > r$. Hence, $|q_N| = N^2 > r$, and thus (q_n) is unbounded. □

4.5 Limit Laws

Proposition 4.16. *Suppose $(q_n)_{n=1}^{\infty}$ is a sequence of rational numbers that converges to a positive $q \in \mathbb{Q}_{>0}$. Then there exists a lower bound $s \in \mathbb{Q}_{>0}$ and a natural number $N \in \mathbb{N}$ such that $q_n > s$ whenever $n \geq N$.*

Proof. Let $\varepsilon = \frac{q}{2}$. Since $q_n \to q$, there exists a natural number $N \in \mathbb{N}$ such that $|q - q_n| < \varepsilon$ whenever $n \geq N$. For such n, $|q - q_n| < \varepsilon$ implies that $q_n > q - \varepsilon$. Hence, $q_n > \frac{q}{2}$ for all $n \geq N$, thus $(q_n)_{n=1}^{\infty}$ has a positive lower bound. □

Proposition 4.17. *Suppose $(q_n)_{n=1}^{\infty}$ is a sequence of nonnegative rational numbers. If $q_n \to q$, then $q \geq 0$.*

Proof. Towards contradiction, suppose $q < 0$. Let $\varepsilon = |q|$. As $q < 0$, $\varepsilon = |q| > 0$. Since $q_n \to q$, for some $N \in \mathbb{N}$, $|q - q_n| < \varepsilon$ for all $n \geq N$. Hence, for such n, $q > q_n - \varepsilon$, and thus $q + |q| > q_n \geq 0$. However, since q is negative, $|q| = -q$, and thus $0 = q + |q| > 0$, a contradiction. Therefore, $q \geq 0$. □

Theorem 4.18. *Suppose $(q_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ are sequences of rational numbers such that $q_n \to q$ and $r_n \to r$. Then the following limit laws hold:*

(a) If $c \in \mathbb{Q}$, the constant sequence $s_n := c$ converges to c.

(b) The sequence $(q_n + r_n)_{n=1}^{\infty}$ converges to $q + r$.

(c) The sequence $(q_n r_n)_{n=1}^{\infty}$ converges to qr.

(d) If $q_n \neq 0$ for all $n \in \mathbb{N}$ and $q \neq 0$, then the sequence $(\frac{1}{q_n})_{n=1}^{\infty}$ converges to $\frac{1}{q}$.

(e) If $q_n \leq r_n$ for all $n \in \mathbb{N}$, then $q \leq r$.

(f) The sequence $(|q_n|)_{n=1}^{\infty}$ converges to $|q|$.

Proof.

(a) Let $\varepsilon > 0$. Set $N = 1$. For all $n \geq N$, $|c - c| = 0 < \varepsilon$. Hence, the sequence $(c, c, \dots)$ converges to c.

(b) Let $\varepsilon > 0$. Since $q_n \to q$, there exists an $N_1 \in \mathbb{N}$ such that $|q - q_n| < \frac{\varepsilon}{2}$ whenever $n \geq N$. Similarly, for some $N_2 \in \mathbb{N}$, $|r - r_n| < \frac{\varepsilon}{2}$ whenever $n \geq N_2$. We will now show that, whenever $n \geq \max(N_1, N_2)$, $|(q + r) - (q_n + r_n)| < \varepsilon$.

$$\begin{aligned} |(q + r) - (q_n + r_n)| &= |(q - q_n) + (r - r_n)| \\ &\leq |q - q_n| + |r - r_n| \end{aligned}$$

Since $n \geq \max(N_1, N_2) \geq N_1$, $|q - q_n| < \frac{\varepsilon}{2}$. Similarly, $n \geq \max(N_1, N_2) \geq N_2$, so $|r - r_n| < \frac{\varepsilon}{2}$. Hence,

$$|(q + r) - (q_n + r_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(c) Let $\varepsilon > 0$. Since $q_n \to q$, by Proposition 4.13, for some $s \in \mathbb{Q}_{>0}$, $|q_n| \leq s$ for all n. Moreover, the convergence of (q_n) implies that there exists a natural number $N_1 \in \mathbb{N}$ such that $|q - q_n| < \frac{\varepsilon}{2|r|+1}$ whenever $n \geq N_1$. Similarly, since $r_n \to r$, there exists a $N_2 \in \mathbb{N}$ such that $|r - r_n| < \frac{\varepsilon}{2s}$ whenever $n \geq N_2$. We will now show that, if $n \geq \max(N_1, N_2)$, $|qr - q_n r_n| < \varepsilon$.

$$\begin{aligned} |qr - q_n r_n| &= |qr - q_n r + q_n r - q_n r_n| \\ &\leq |qr - q_n r| + |q_n r - q_n r_n| \\ &= |q - q_n||r| + |r - r_n||q_n| \\ &\leq |q - q_n||r| + |r - r_n| \cdot s \end{aligned}$$

Since $n \geq \max(N_1, N_2) \geq N_1$, $|q - q_n| < \frac{\varepsilon}{2|r|+1}$. Similarly, $n \geq N_2$, so $|r - r_n| < \frac{\varepsilon}{2s}$. Hence,

$$\begin{aligned} |qr - q_n r_n| &< \frac{\varepsilon}{2|r|+1} \cdot |r| + \frac{\varepsilon}{2s} \cdot s \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore, the product sequence $(q_n r_n)_{n=1}^{\infty}$ converges to qr.

(d) Without loss of generality, assume $q > 0$ (if $q < 0$, negating all q_n via (c) makes q positive and generalizes this result). Let $\varepsilon > 0$. As $q > 0$, by Proposition 4.16, there exists a positive $s \in \mathbb{Q}_{>0}$ and $N_1 \in \mathbb{N}$ such that $q_n > s$ whenever $n \geq N_1$. Moreover, since $q_n \to q$, there exists an $N_2 \in \mathbb{N}$ such that $|q - q_n| < |q| s\varepsilon$ whenever $n \geq N_2$. We will now show that $|\frac{1}{q} - \frac{1}{q_n}| < \varepsilon$ whenever $n \geq \max(N_1, N_2)$. For such n, since $n \geq N_2$, we have

$$\begin{aligned} \left|\frac{1}{q} - \frac{1}{q_n}\right| &= \left|\frac{q_n - q}{q q_n}\right| \\ &= \frac{1}{|q||q_n|}|q_n - q| \\ &< \frac{1}{|q||q_n|}(|q| s\varepsilon) \\ &= \frac{s}{|q_n|} \cdot \varepsilon \end{aligned}$$

Since $n \geq N_1$, $s < q_n$, and hence

$$\left|\frac{1}{q} - \frac{1}{q_n}\right| < \varepsilon$$

(e) From the previous limit laws, it follows that the sequence $(r_n - q_n)_{n=1}^{\infty}$ converges to $r - q$. Since $q_n \leq r_n$ for all n, each element of this sequence is nonnegative. By Proposition 4.17, the limit $r - q$ must be nonnegative as well. Therefore, $q \leq r$.

(f) Let $\varepsilon > 0$. As $q_n \to q$, for some $N \in \mathbb{N}$, $|q - q_n| < \varepsilon$ for all $n \geq N$. For such n, by the reverse triangle inequality, $||q| - |q_n|| \leq |q - q_n| < \varepsilon$. Hence, $(|q_n|)_{n=1}^{\infty}$ converges to $|q|$.

□

4.6 Cauchy Sequences

Remark. We are trying to use convergent sequences to construct the real numbers. However, only sequences that approach rational numbers converge. That is, if a sequence limits to a "gap" in the rationals (ie. an *irrational* number), it would *not* converge, since there would be no rational q to which the sequence can limit. Cauchy sequences will formalize the idea that such sequences "stabilize," even if they don't necessarily converge in $\mathbb{Q}$.

Definition 4.19. A sequence $(q_n)_{n=1}^{\infty}$ is **Cauchy** if, for all $\varepsilon \in \mathbb{Q}_{>0}$, there exists a positive $N \in \mathbb{N}$ such that, for all $m, n \geq N$, $|q_m - q_n| < \varepsilon$.

Proposition 4.20. *Equivalently, a sequence $(q_n)_{n=1}^{\infty}$ is **Cauchy** if (and only if) for all $\varepsilon \in \mathbb{Q}_{>0}$, there exists a positive $N \in \mathbb{N}$ such that, for all $n \geq N$, $|q_n - q_N| < \varepsilon$.*

Proof. Clearly, all Cauchy sequences satisfy this condition, as it is equivalent to letting m to be *exactly* N. To prove the other direction, let $\varepsilon > 0$. Then, by our new condition, for some $N \in \mathbb{N}$, $|q_n - q_N| < \frac{\varepsilon}{2}$ for all $n \geq N$. Hence, for $m, n \geq N$, both $|q_m - q_N| < \frac{\varepsilon}{2}$ and $|q_n - q_N| < \frac{\varepsilon}{2}$. By the triangle inequality, $|q_m - q_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and thus $(q_n)_{n=1}^{\infty}$ is Cauchy. □

Proposition 4.21. *If a sequence $(q_n)_{n=1}^{\infty}$ converges, it is Cauchy.*

Proof. Suppose $q_n \to q$ and let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|q - q_n| < \frac{\varepsilon}{2}$ for all $n \geq N$. Hence, whenever $m, n \geq N$, by the triangle inequality, $|q_n - q_m| \leq |q - q_n| + |q - q_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

Example 4.22. Let $q_n := \max\{\frac{a}{10^n} : a \in \mathbb{N} \text{ and } (\frac{a}{10^n})^2 \leq 2\}$ be the sequence "approximating $\sqrt{2}$." Then $(q_n)_{n=1}^{\infty}$ is Cauchy but does *not* converge in $\mathbb{Q}$.

Proof. We will prove each of the two claims independently.

(i) $(q_n)_{n=1}^{\infty}$ does not converge in $\mathbb{Q}$.
For contradiction, suppose $q_n \to q$. By construction, for each $n \in \mathbb{N}$, $q_n^2 \leq 2 < (q_n + \frac{1}{10^n})^2$. As $(\frac{1}{10^n})_{n=1}^{\infty}$ converges to 0, by the limit laws, $(q_n + \frac{1}{10^n})_{n=1}^{\infty}$ converges to q. Moreover, the limit laws imply that $q_n^2 \to q^2$ and $(q_n + \frac{1}{10^n})^2 \to q^2$. However, since limits preserve ordering, we obtain $q^2 \leq 2 \leq q^2$. Hence, $q^2 = 2$ for a rational $q \in \mathbb{Q}$, a contradiction. Thus, the sequence $(q_n)_{n=1}^{\infty}$ does not converge in $\mathbb{Q}$.

(ii) $(q_n)_{n=1}^{\infty}$ is Cauchy.
Let $\varepsilon > 0$. By the Archimedean Property, there exists a natural number $N > \frac{1}{\varepsilon}$. We will show that, for all $m, n \geq N$, $|q_m - q_n| < \varepsilon$. Without loss of generality, assume $m \geq n$. Note that, by construction, $q_n \leq q_{n+1} \leq q_n + \frac{9}{10^{n+1}}$ for all n. We now calculate as follows:

$$\begin{aligned}
|q_m - q_n| &= |(q_m - q_{m-1}) + (q_{m-1} - q_{m-2}) + \cdots + (q_{n+1} - q_n)| \\
&\leq |q_m - q_{m-1}| + |q_{m-1} - q_{m-2}| + \cdots + |q_{n+1} - q_n| \\
&\leq \frac{9}{10^m} + \frac{9}{10^{m-1}} + \cdots + \frac{9}{10^{n+1}} \\
&\leq \left(\frac{9}{10^{n+1}}\right)\left(\frac{1}{10^{m-n-1}} + \frac{1}{10^{m-n-2}} + \cdots + 1\right) \\
&= \left(\frac{9}{10^{n+1}}\right)\left(\frac{1 - \frac{1}{10^{m-n}}}{1 - \frac{1}{10}}\right) \\
&< \frac{9}{10^{n+1}} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{9}{10^{n+1}} \cdot \frac{10}{9} = \frac{1}{10^n} \\
&< \frac{1}{n} \leq \frac{1}{N} < \varepsilon.
\end{aligned}$$

□