Def | An EQUAVALENCE RELATION on X is a relation $\sim \subseteq X \times X$ that is.

REFLEXTIVE, for all $x \in X$ $x \sim x$

SYMMETRIC for all $x \in X$, if $x \sim y$, then $y \overset{\text{is relatied to}}{\sim} x$

Ex: " is a sibling of " → symmetric

" is a brother of " → not symmetric

TRANSITIVE for all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$

Ex. " is ancestor of " → yes

" is a parent of " → no

Def | Equivalent class $[x]_\sim := \{y \in X : x \sim y\}$ Also written as $[x]$.

Property | let $\sim$ be an equivalent relation on X and, $x, y \in X$

(a) if $x \sim y$, then $[x] = [y]$ $\qquad [x] = \{x, y\}$

(b) if $x \nsim y$, then $[x] \cap [y] = \phi$ $\qquad [y] = \{x, y\}$

Proof | (a)

($\subseteq$) Suppose $z \in [x]$, then by def'n $x \sim z$

Given that $x \sim y$

By symmetry of equiv. rel., $y \sim x$, $x \sim z$

By transitivity of equiv rel, $y \sim x \sim z$

$\therefore y \sim z$

And by def'n of equiv. class, $z \in [y]$

$\therefore [x] \subseteq [y]$

($\supseteq$) Suppose $z \in [y]$

By symmetry of equiv. rel., we also know that $[y] \subseteq [x]$

Overall, $[x] = [y]$

(b) (proof by contradiction)

Suppose $x \nsim y$, and $a \in [x] \cap [y]$

Then, by def'n, $x \sim a$ and $y \sim a$

By symmetry of equiv. rel., $x \sim a$

Therefore, $x \sim a \sim y$

Then, $x \sim y$, which contradicts with the assump. that $x \nsim y$

Then, $a \notin [x] \cap [y]$

We conclude that $[x] \cap [y]$ has no element if $x \neq y$

Rational Numbers ($\mathbb{Q}$)

We want to construct #s that represent "dividing" an integer a into b parts

On the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, we define an equi. rel. as follows:

For $(a,b), (c,d) \in \mathbb{Z} \times \mathbb{Z}^*$,

we declare $(a,b) \sim (c,d)$, when $a \cdot d = b \cdot c$

Def| The rational numbers are $\mathbb{Q} := \{\text{equi. class of } \sim\} = \mathbb{Z} \times \mathbb{Z}^* / \sim$

equiv. class of $\sim$

Def| $[(a,b)] + [(c,d)] = [(ad+bc, bd)]$

This "+" is well-defined

Proof| Given that $(a,b) \sim (a',b')$, $(c,d) \sim (c',d')$,

WTS $(ad+bc, bd) \sim (a'd'+b'c', b'd')$

That is, by def'n of "$\sim$", WTS $(ad+bc)b'd' = (a'd'+b'c')bd$

$\because$ By def'n of "$\sim$", $ab' = ba'$, $cd' = dc'$

$$
\begin{aligned}
\therefore (ad+bc)b'd' &= adb'd' + bcb'd' \\
&= a'bdd' + bb'dc' \\
&= (a'd' + b'c')bd
\end{aligned}
$$

$\therefore (ad+bc, bd) \sim (a'd'+b'c', b'd')$

But the def'n of $\mathbb{Q}$ is still insufficient e.g. these is no rational q with $q^2 = 2$

Proof: (The proof is by contradiction)

Suppose there's such a rational number q

Then we can write $q = \frac{m}{n}$, We can assume that $n > 0$ and as small as possible.

Then $q^2 = 2$ implies $m^2 = 2n^2$

So, m^2 is even. This, in turn, implies m is even (b/c if m were odd ($m = 2k+1$, for some $k \in \mathbb{Z}$),

so $m^2 = (2k+1)(2k+1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is still an odd

$\therefore$ m^2 is even $\Rightarrow$ m is even as well.)

Hence, $m = 2k$ for some $k \in \mathbb{Z}$

So, $4k^2 = 2n^2 \Rightarrow 2k^2 = n^2$

$\therefore$ n^2 is even $\Rightarrow$ n is even

$\therefore$ m, n are both even

$\therefore$ they have the same factor 2

$\therefore$ n is not as small as possible (m, n could be divided by 2), which contradicts with the assumption that n is as small as possible.

However,

Notice that we can "approximate" a positive # whose square is 2 using rational #

eg:

rational #	1.4	1.41	1.414
	$= 1 + \frac{4}{10}$	$1 + \frac{4}{10} + \frac{1}{100}$	$1 + \frac{4}{10} + \frac{1}{100} + \frac{4}{1000}$
	$1.4^2 = 1.96$	$1.41^2 = 1.9881$	$1.414^2 = 1.999396$

Therefore, We will construct $\mathbb{R}$ using sequence of rational #s ($\mathbb{Q}$)

SEQUENCES

Def: A sequence in $\mathbb{Q}$ (i.e. of rational #s) is a func. $q: \mathbb{Z}_{>0} \to \mathbb{Q}$

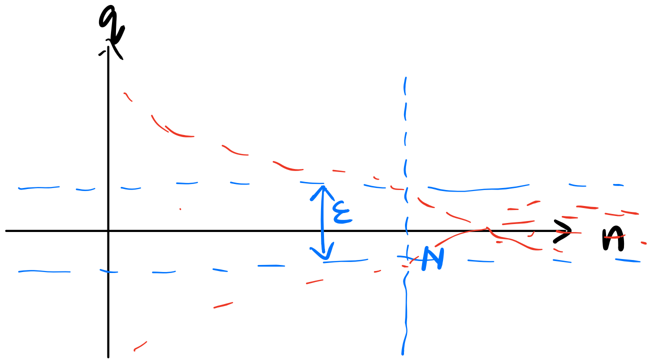
We write q_n for $q(n)$

and $(q_n)_{n=1}^{\infty}$ [or $(q_n)_n$, (q_n), $\{q_n\}_{n=1}^{\infty}$] for the sequence

Ex: If $q_n = \frac{1}{n}$, then $\underset{q(1)}{q_1 = \frac{1}{1}}$, $\underset{q(2)}{q_2 = \frac{1}{2}}$, $\underset{q(3)}{q_3 = \frac{1}{3}}$

Def A sequence (q_n) in $\mathbb{Q}$ CONVERGES to $q \in \mathbb{Q}$ if

$\underbrace{\forall \varepsilon \in \mathbb{Q}_{>0}}_{\text{for all positive rational \#}}, \underbrace{\exists N \in \mathbb{Z}_{>0}}_{\text{there exists a positive integer } N}, \underbrace{\forall n \geq N}_{\text{for all } n \geq N \text{ (all } n \text{ beyond } N)} \underbrace{(|q - q_n| < \varepsilon)}_{\text{diff between } q \text{ and } q_n \text{ is small}}$



We then call q is a limit of (q_n) and we write $q_n \to q$

(or $\lim_{n\to\infty} q_n = q$. $\lim q_n = q$, $q_n \xrightarrow{n\to\infty} q$

Ex $q_n = \frac{1}{n}$, let's show $q_n \to 0$

Proof Suppose $\varepsilon \in \mathbb{Q}_{>0}$

Then WTS $|0 - \frac{1}{n}| < \varepsilon$ for $n \geq N$, where N is TBD

That is, $\frac{1}{n} < \varepsilon$ for $n \geq N$, where N is TBD

Or $\frac{1}{\varepsilon} < n$ for $n \geq N$, where N is TBD

Since $\varepsilon \in \mathbb{Q}_{>0}$ $\therefore \frac{1}{\varepsilon} \in \mathbb{Q}_{>0}$

$\therefore \frac{1}{\varepsilon}$ can also be written as $\frac{1}{\varepsilon} = \frac{a}{b}$, $a, b \in \mathbb{Z}_{>0}$

It's obvious that, $a \geq \frac{a}{b}$ for $a, b \in \mathbb{Z}_{>0}$

Let $N = a+1$, then we can have $N > a$

Then, we have $n \geq N > a \geq \frac{a}{b} = \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon}$

Therefore, we showed that $n \geq N$ implies $|0 - \frac{1}{n}| < \varepsilon$ for $N = a+1$, $a \in \mathbb{Z}_{>0}$