lecture 2

II. Sets

① Definition: A set is a collection of elements. If x is an element of a set A, we write $x \in A$.

② Definition: Two sets A and B are said to be equal if they have the same elements. We write: $A = B$

A is said to be a subset of B if every element of A is an element of B. We write: $A \subseteq B$

Thus, $A = B$ means $A \subseteq B$ and $B \subseteq A$

③ Notation: We can specify a set by listing its elements. e.g. $A = \{1, 2, 3\}$

More generally, if A is a set and $P(x)$ is some formula, we can construct $\{x \in A : P(x)\}$

e.g. $\{x \in \mathbb{Z} : -1 \leq x \leq 1\} = \{-1, 0, 1\}$

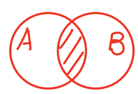
if f is some function, we can construct $\{f(x) : x \in A\}$

e.g. $\{2x : x \in \mathbb{Z}\} = \{0, 2, -2, 4, -4, \cdots\}$

④ Definition: The empty set is the set $\emptyset := \{\ \}$

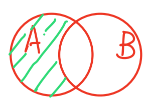
⑤ Definition: Given sets A and B, we can form their union $A \cup B := \{x : x \in A \vee x \in B\}$

intersection $A \cap B := \{x : x \in A \wedge x \in B\}$



difference $A \setminus B := \{x : x \in A \wedge x \notin B\}$

$\neg(x \in B)$



e.g. $A = \{1, 2, 3\}$

$B = \{2, 3, 4\}$

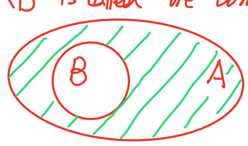
$A \cup B = \{1, 2, 3, 4\}$

$A \cap B = \{2\}$

$A \setminus B = \{1\}$

Based on difference

⑥ Definition: When $B \subseteq A$, the difference $A \setminus B$ is called the complement of B in A; if A is clear from the context, it is often denoted B^c



e.g. $A = \mathbb{Z}$, $B = \{2, 3, 4\}$

$A \setminus B = \{\cdots -1, 0, 1, 5, 6, \cdots\} = B^c$

⑦ Definition: The power set of a set A is the set of all subsets of A, denoted $P(A)$

e.g. $A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Note: 2 sets are equal if they have the same element(s) ⇒ sequence/duplicate doesn't matter above.

⑧ Definition: The (Cartesian) product of a set A and a set B is $A \times B := \{\underbrace{(a, b)}_{} : a \in A, b \in B\}$

ordered pair, so order does matter here.

e.g. $A = \{1, 2, 3\}$ $B = \{2, 3, 4\}$

$A \times B = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$

Example: Let A, B be sets. Show that $P(A) \cup P(B) \subseteq P(A \cup B)$ Example Proof

Suppose $C \in P(A) \cup P(B)$, then $C \in P(A)$ or $C \in P(B)$

- If $C \in P(A)$, then, by definition, $C \subseteq A$

  Hence, $C \subseteq A \cup B$, so $C \subseteq P(A \cup B)$

- If $C \in P(B)$, then, by definition, $C \subseteq B$

  Hence, $C \subseteq A \cup B$, so $C \subseteq P(A \cup B)$

- This shows $P(A) \cup P(B) \subseteq P(A \cup B)$

Exercises:

① If $B \subseteq C$, then $A \times B \subseteq \underbrace{A \times C}_{\{(a,b) : a \in A, b \in C\}}$

Suppose $x \in A \times B$. Then, by definition, $x = (a, b)$, where $a \in A$ and $b \in B$

Since $B \subseteq C$ by construction, then we know $b \in C$

Hence, by definition, $x \in A \times C$.

This shows that $A \times B \in A \times C$

② $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(⇒) Suppose $x \in A \times (B \cup C)$, which means

by definition, $x = (a, b)$, where $a \in A$ and $b \in B \cup C$

so then, $b \in B$ or $B \in C$

- For $x = (a, b)$, $a \in A$ and $b \in B$

  we can write this situation as $x = \{(a, b), a \in A, b \in B\}$

(⇐) Suppose $x \in (A \times B) \cup (A \times C)$, which means

by definition, $x \in A \times B$ or $x \in A \times C$

- For $x \in A \times B$, we can define it as: $x = (a, b)$ where $a \in A$, $b \in B$

  then we know that $x = \{(a, b), a \in A, b \in B\}$

  this belongs to $x' = \{(a', b'), a' \in A, b' \in B \cup C\} = A \times (B \cup C)$

so we proved that $x \subseteq A \times B$

- For $x = (a, b)$, $a \in A$, $b \in C$

  we can write this situation as $x = \{(a,b), a \in A, b \in C\}$

  so we proved that $x \subseteq A \times C$

- Since $(A \times B) \cup (A \times C)$, we know that the relationship between satisfication of $(A \times B) \cup (A \times C)$ is OR, which means one element belongs to either $A \times B$ or $A \times C$ belongs to $(A \times B) \cup (A \times C)$

- We just proved two situations that $x \in A \times B$ or $x \in A \times C$

- We proved that $\boxed{A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)}$

so then $x \subseteq A \times (B \cup C)$

- For $x \in A \times C$, we can define it as: $x = (a, b)$ where $a \in A$, $b \in C$

  then we know that $x = \{(a, b), a \in A, b \in C\}$

  this belongs to $x' = \{(a, b'), a \in A, b' \in B \cup C\} = A \times (B \cup C)$

  so then, $x \subseteq A \times (B \cup C)$

- For both of the situations we proved that $x \subseteq A \times (B \cup C)$,

  hence, we get $\boxed{(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)}$

- Since we proved $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$, then we know that $(A \times B) \cup (A \times C) = A \times (B \cup C)$

③ $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

($\Rightarrow$) Suppose $x \in X \setminus (A \cup B)$

Then, by definition, we know that $x \in X$, but $x \notin (A \cup B)$

this means $x \in X$, but $x \notin A$ and $x \notin B$

So then, we know that $x \in X \setminus A$ and $x \in X \setminus B$

This satisfy $x \in (X \setminus A) \cap (X \setminus B)$

($\Leftarrow$) Suppose $x \in (X \setminus A) \cap (X \setminus B)$

Then, by definition $x \in X \setminus A$ and $x \in X \setminus B$

So, we know, $x \in X$, $x \notin A$ and $x \in X$, $x \notin B$

Then, we know, $x \in X$ but $x \notin A$ and $x \notin B$

Then, $x \in X \setminus (A \cup B)$

Since we proved from both direction that $X \setminus (A \cup B) \subseteq / \supseteq (X \setminus A) \cap (X \setminus B)$, we conclude that $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

④ If $A \subseteq B$, then $A \setminus C \subseteq B \setminus C$ $\quad P \Rightarrow Q = \neg P \vee Q = \neg(P \wedge \neg Q) = \neg Q \Rightarrow \neg P$

Suppose $x \in A \setminus C$, then we know $x \in A$ but $x \notin C$

Since $A \subseteq B$, we know that $x \in B$

Then, we know $x \in B$ but $x \notin C$, which means $x \in B \setminus C$

Since we proved that $x \in A \setminus C$ implies $x \in B \setminus C$

Then, we know that $x \subseteq A \setminus C \subseteq B \setminus C$

Hence, we can conclude that if $A \subseteq B$, then $A \setminus C \subseteq B \setminus C$

II. Relations and Functions

① Definition: Let X and Y be sets. A relation between X and Y is a set $R \subseteq X \times Y$. If $(x, y) \in R$, we write xRy. The set X is called the domain and the set Y is called the codomain of the relation.

e.g. $X = \{1, 2, 3\}$ $\quad Y = \{2, 3, 4\}$

example of relation

$\boxed{\leq} = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)\}$
$\quad 1 \leq 2 \quad 1 \leq 3 \quad 1 \leq 4 \quad \ldots$

$\boxed{>} = \{(3,2)\}$
$\quad 3 > 2$

$\boxed{=} = \{(2,2), (3,3)\}$
$\quad 2 = 2$
same $\hookrightarrow$ $(2,2) \in =$

not all the relations have special symbles e.g. y is double of x.

② Definition: A function from X to Y is a relation $f \subseteq X \times Y$, s.t. for all $x \in X$, there exists a unique (i.e. exactly one) $y \in Y$ with xfy

If xfy, we write $y=f(x)$. If there are multiple y related to x, we wouldn't know which $f(x)$ refer to.

Notation: $f: X \to Y$ means f is a function from X to Y.

③ Definition: Let $f: X \to Y$ be a function:

- f is injective / injection / one-to-one if for all $x, x' \in X$, if $f(x) = f(x')$, then $x = x'$
- f is surjective / surjection / onto if for all $y \in Y$, there exists an $x \in X$ s.t. $y = f(x)$
- f is bijective / bijection / 1-to1 & onto if f is injective and surjective

④ Definition: The inverse of a relation $R \subseteq X \times Y$ is the relation $R^{-1} := \{(y, x) \in Y \times X : xRy\}$
same meaning as $yR^{-1}x$

e.g. $X = \{1, 2, 3\}$ $Y = \{2, 3, 4\}$

$\leq \; = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)\}$

$\leq^{-1} = \{(2,1), (3,1), (4,1), (2,2), (3,2), (4,2), (3,3), (4,3)\}$

⑤ Prop: Let $f: X \to Y$ be a function. Then f^{-1} is a function if and only if f is a bijection

⑥ Definition: Let $f: X \to Y$ be a function. The image of $A \subseteq X$ under f is $f(A) := \{f(x) : x \in A\}$

The preimage of $B \subseteq Y$ under f is $f^{-1}(B) := \{x \in X : f(x) \in B\}$
In this notation for preimage, we do NOT assume that f^{-1} is a function. BUT if f^{-1} is a function, then $f^{-1}(B) = (f^{-1})(B)$

e.g. Let $X = \mathbb{Z}$, $Y = \mathbb{Z}$, $f(x) := x^2 = x \cdot x$

$f(\{1, 2, -3\}) = \{1, 4, 9\}$ ⇐ image

$f^{-1}(\{1, 4\}) = \{1, 2, -1, -2\}$ ⇐ preimage (this prove that preimage can be a non-function)

e.g. Let $f: X \to Y$ be a function. If $B_1, B_2 \subseteq Y$, then $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$ Example Proof

($\Rightarrow$) Suppose that $x \in f^{-1}(B_1 \cup B_2)$.

Then, by definition, $f(x) \in B_1 \cup B_2$

- If $f(x) \in B_1$, then $x \in f^{-1}(B_1)$ by definition
- If $f(x) \in B_2$, then $x \in f^{-1}(B_2)$ by definition

Hence, $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$, which shows that $f^{-1}(B_1 \cup B_2) \subseteq f^{-1}(B_1) \cup f^{-1}(B_2)$

($\Leftarrow$) Suppose $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.

Then $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, meaning that $f(x) \in B_1$ or $f(x) \in B_2$

Hence, $f(x) \in B_1 \cup B_2$, which by definition means that $x \in f^{-1}(B_1 \cup B_2)$

So we conclude that $f^{-1}(B_1) \cup f^{-1}(B_2) \subseteq f^{-1}(B_1 \cup B_2)$

- Therefore, $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

⑦ Definition: A equivalent relation on X is a relation $\sim \subseteq X \times X$ that is:

- reflexive: for all $x \in X$, $x \sim x$

To prove an equivalent relation, we need to prove if transitive, reflexive, symmetric satisfied.

- symmetric: for all $x, y \in X$, if $x \sim y$, then $y \sim x$

  e.g. Consider one girl one boy situation:

  "is a sibling of" is symmetric

  "is a brother of" is NOT symmetric

- transitive: for all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$

  e.g. Consider 1 child, 1 parent, 1 grandparent situation:

  "is an ancestor of" is transitive

  "is a parent of" is NOT transitive

⑧ Definition: Let $\sim$ be an equivalent relation on X. The equivalent class of $x \in X$ is $[x]_\sim := \{y \in X : x \sim y\}$

Also written as $[x]$

e.g. X = people in MATH 131A $\quad X = \{x, y, z\}$

$x \sim y$ means x and y have the same birth month (Feb.) $\quad$ $z \sim a$ means z and a have the same birth month (August)

This is an equivalent equation since $x \sim x$, $x \sim y$, $x \not\sim z$

$[x]_\sim = \{x, y\}$ $\quad$ $[x]_\sim$ means the set that include all the equivalent relation elements w/ x.

e.g. in MATH 131A, all people born in June are A, B, C, then $[A]_\sim = \{A, B, C\}$

$[z]_\sim = \{z, a\}$