MATH 131A	August 5, 2024

Lecture 1: Logic

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1.1 Logic

Definition 1.1. A **statement** is a sentence that is true (T) or false (F).

Example 1.2. Let P denote the statement "2 is even" and Q denote the statement "2 is odd". Then P is true, whereas Q is false.

Definition 1.3. The **conjunction** of a statement P and a statement Q is the statement "P and Q" and is denoted $P \wedge Q$.

The truth value (T or F) of $P \wedge Q$ depends on those of P and Q, as summarized by the following **truth table**.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.4. The **disjunction** of a statement P and a statement Q is the statement "P or Q (or both)" and is denoted $P \vee Q$.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 1.5. The **negation** of a statement P is the statement "it is not the case that P" and is denoted $\neg P$.

P	$\neg P$
T	F
F	T

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Proposition 1.6 (De Morgan's laws)**.** *Let P and Q be statements. Then*

$$\neg(P \wedge Q) = \neg P \vee \neg Q, \tag{1.1}$$
$$\neg(P \vee Q) = \neg P \wedge \neg Q, \tag{1.2}$$

where the $=$ sign means that the statements on both sides of the sign have the same truth value for all possible truth values of P and Q; this relation is called ***logical equivalence****.*

Proof. Exercise. □

Definition 1.7. Given statements P and Q, the **conditional** or **implication** $P \Rightarrow Q$ is the statement "if P, then Q" (equivalently, "Q if P", "P only if Q", etc.).

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Proposition 1.8. *Let P and Q be statements. Then*

$$P \Rightarrow Q = \neg P \vee Q. \tag{1.3}$$

Proof. Exercise. □

Proposition 1.9. *Let P and Q be statements. Then*

$$P \Rightarrow Q = \neg Q \Rightarrow \neg P. \tag{1.4}$$

The implication $\neg Q \Rightarrow \neg P$ is called the ***contrapositive*** *of the implication $P \Rightarrow Q$.*

Proof. Exercise. □

Definition 1.10. Given statements P and Q, the **biconditional** or **biimplication** $P \Leftrightarrow Q$ is the statement "P if and only if Q".

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Proposition 1.11. *Let P and Q be statements. Then*

$$P \Leftrightarrow Q = (P \Rightarrow Q) \wedge (Q \Rightarrow P). \tag{1.5}$$

The implication $Q \Rightarrow P$ is called the ***converse*** *of the implication $P \Rightarrow Q$.*

Proof. Exercise. □

Proposition 1.12. *Let P, Q, and R be statements. Then*

$$P \wedge Q = Q \wedge P, \qquad P \vee Q = Q \vee P, \tag{1.6}$$
$$P \wedge (Q \wedge R) = (P \wedge Q) \wedge R, \qquad P \vee (Q \vee R) = (P \vee Q) \vee R, \tag{1.7}$$
$$P \wedge \mathrm{T} = P, \qquad P \vee \mathrm{F} = P, \tag{1.8}$$
$$P \wedge \neg P = \mathrm{F}, \qquad P \vee \neg P = \mathrm{T}, \tag{1.9}$$
$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R), \qquad P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R). \tag{1.10}$$

Proof. Exercise. □

More generally, we can consider formulas or sentences with "free variables", called "open formulas" or "open sentences", that become statements once these variables are "bound", forming "closed formulas" or "closed sentences" (we will not define these terms formally in this course).

Example 1.13. Let $P(n)$ denote the open sentence "n is even". The free variable n can be bound to form statements such as "$P(2)$", "$P(3)$", "$P(n)$, where n is 2 or 4", "for all integers n, $P(n)$ or it is not the case that $P(n)$", or "there exists an integer n such that $P(n)$".

Definition 1.14. Universal quantification is asserting something for *all* values of a variable (using phrases such as "for all" or "for every") and is denoted by $\forall$.

Definition 1.15. Existential quantification is asserting something for *some* value of a variable (using phrases such as "for some" or "there exists") and is denoted by $\exists$.

Example 1.16. Let $P(n)$ denote the open sentence "n is even". The statement "for all integers n, $P(n)$ or it is not the case that $P(n)$" can be written symbolically as $\forall n\,(n \in \mathbb{Z} \implies P(n) \vee \neg P(n))$. Similarly, the statement "there exists an integer n such that $P(n)$" can be written symbolically as $\exists n\,(n \in \mathbb{Z} \wedge P(n))$.

We can abbreviate the statement $\forall n\,(n \in \mathbb{Z} \implies P(n) \vee \neg P(n))$ using ***bounded*** **universal quantification** as $\forall n \in \mathbb{Z}\,(P(n) \vee \neg P(n))$. Similarly, we can abbreviate the statement $\exists n\,(n \in \mathbb{Z} \wedge P(n))$ using ***bounded*** **existential quantification** as $\exists n \in \mathbb{Z}\,(P(n))$.

Proposition 1.17 (De Morgan's laws)**.** *Let $P(x)$ be an open sentence. Then*

$$\neg \forall x\,(P(x)) = \exists x\,(\neg P(x)), \tag{1.11}$$
$$\neg \exists x\,(P(x)) = \forall x\,(\neg P(x)). \tag{1.12}$$

Remark. De Morgan's laws also hold for bounded quantification.

Proof. Exercise. □