Linear maps and matrices

Let $\mathbb{F}$ be a field and let V and W be vector spaces over $\mathbb{F}$ with respective bases $B_V := \{v_j\}_{j=1}^n$ and $B_W := \{w_i\}_{i=1}^m$.

Matrices

- The **matrix** of $T \in L(V,W)$ with respect to B_V and B_W is

$$ {}_{B_W}[T]_{B_V} := \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix} \in \mathbb{F}^{m\times n}, \quad \text{where } \alpha_{ij} \in \mathbb{F} \text{ are such that } Tv_j = \sum_{i=1}^m \alpha_{ij} w_i \,. $$

- The **matrix** of $v \in V$ with respect to B_V is

$$ \begin{aligned} {}_{B_V}[v] &:= {}_{B_V}[\Phi_v]_{\{1_{\mathbb{F}}\}} \in \mathbb{F}^{n\times 1}, && \text{where } \Phi_v \in L(\mathbb{F},V) \text{ is defined by } \Phi_v(\alpha) := \alpha v \\ &= \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}, && \text{where } \beta_j \in \mathbb{F} \text{ are such that } v = \sum_{j=1}^n \beta_j v_j \,. \end{aligned} $$

  Thus,

$$ {}_{B_W}[T]_{B_V} = \begin{bmatrix} {}_{B_W}[Tv_1] & \cdots & {}_{B_W}[Tv_n] \end{bmatrix}. $$

Matrix operations

- For $T_1, T_2 \in L(V,W)$ and $\alpha \in \mathbb{F}$, we define

$$ \begin{aligned} {}_{B_W}[T_1]_{B_V} + {}_{B_W}[T_2]_{B_V} &:= {}_{B_W}[T_1 + T_2]_{B_V} \,, \\ \alpha \cdot {}_{B_W}[T_1]_{B_V} &:= {}_{B_W}[\alpha T_1]_{B_V} \,. \end{aligned} $$

  Thus, the map $T \mapsto {}_{B_W}[T]_{B_V}$ is an $\mathbb{F}$-vector space isomorphism from $L(V,W)$ to $\mathbb{F}^{m\times n}$.

- Suppose that U is a vector space over $\mathbb{F}$ with basis $B_U := \{u_k\}_{k=1}^p$. For $S \in L(U,V)$ and $T \in L(V,W)$, we define

$$ {}_{B_W}[T]_{B_V}\, {}_{B_V}[S]_{B_U} := {}_{B_W}[T \circ S]_{B_U} \,. $$

  In particular, since $\Phi_{Tv} = T \circ \Phi_v$ for all $v \in V$, we have

$$ {}_{B_W}[Tv] = {}_{B_W}[T]_{B_V}\, {}_{B_V}[v] \,. $$

  Moreover, the map $T \mapsto {}_{B_V}[T]_{B_V}$ is an $\mathbb{F}$-vector space isomorphism from $L(V)$ to $\mathbb{F}^{n\times n}$ that also preserves multiplication and the multiplicative identity (otherwise known as a “unital $\mathbb{F}$-algebra isomorphism”).