Monday, Week 10

Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be finite-dimensional inner product spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definitions

- **Adjoint**: The adjoint of a map $T \in \mathcal{L}(V, W)$ is the unique linear map $T^* \in \mathcal{L}(W, V)$ such that $\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V \quad \text{for all } v \in V, w \in W.$
- **Normal Operator**: $T \in \mathcal{L}(V)$ is *normal* if $T^*T = TT^*$.
- **Self-adjoint Operator**: $T \in \mathcal{L}(V)$ is *self-adjoint* if $T = T^*$.
- **Unitary Operator**: $T \in \mathcal{L}(V, W)$ is *unitary* if:
  $T^*T = I_V \quad \text{and} \quad TT^* = I_W$.
  Equivalently, $T^* = T^{-1}$.
- **Real and Imaginary Parts**: For any $T \in \mathcal{L}(V)$, we define:
  $\text{Re}(T) := \frac{T+T^*}{2}, \quad \text{Im}(T) := \frac{T-T^*}{2i}$.

Remark: When $V = W$, unitary operators are always normal.

Proposition

Let $T \in \mathcal{L}(V, W)$. The following statements are equivalent:

(a) $T^*T = I_V$

(b) $\langle Tv, Tv' \rangle_W = \langle v, v' \rangle_V$ for all $v, v' \in V$

(c) $\|Tv\|_W = \|v\|_V$ for all $v \in V$

Assume $T^*T = I_V$. We want to show that for all $v, v' \in V$, $\langle Tv, Tv' \rangle_W = \langle v, v' \rangle_V$.
Starting with the left-hand side:

$$\langle Tv, Tv' \rangle_W = \langle T^*Tv, v' \rangle_V = \langle I_V v, v' \rangle_V = \langle v, v' \rangle_V.$$

So (a) implies (b).
Assume $\langle Tv, Tv' \rangle_W = \langle v, v' \rangle_V$ for all $v, v' \in V$.
Let $v = v'$. Then:

$$\langle Tv, Tv \rangle_W = \langle v, v \rangle_V \Rightarrow \|Tv\|_W^2 = \|v\|_V^2 \Rightarrow \|Tv\|_W = \|v\|_V.$$

Hence, (b) implies (c).
Assume $\|Tv\|_W = \|v\|_V$ for all $v \in V$.
Then:

$$\langle Tv, Tv \rangle_W = \|Tv\|_W^2 = \|v\|_V^2 = \langle v, v \rangle_V.$$

Let's compute:

$$\langle Tv, Tv\rangle_W = \langle T^*Tv, v\rangle_V \Rightarrow \langle (T^*T - I_V)v, v\rangle_V = 0 \text{ for all } v \in V.$$

Since $T^*T - I_V$ is self-adjoint (from Week 9), the inner product $\langle (T^*T - I_V)v, v\rangle_V = 0$ for all $v \in V$ implies that $T^*T - I_V = 0$, hence:

$$T^*T = I_V.$$

So (c) implies (a).

Proposition

Let $T \in \mathcal{L}(V)$. Then

$$T \text{ is unitary} \iff T \text{ is normal and } \mathrm{Re}(T)^2 + \mathrm{Im}(T)^2 = I.$$

6/3:

RECAP:

Let V, W inner product spaces. $T \in L(V,W)$.
We say T is "unitary" if:

$$T^* = T^{-1}$$

$$(TT^* = I_W \quad T^*T = I_V)$$

equivalently:

1. for all $v, v' \in V$
   $\langle Tv, Tv' \rangle = \langle v, v' \rangle$
2. for all $v \in V$, $\|Tv\| = \|v\|$

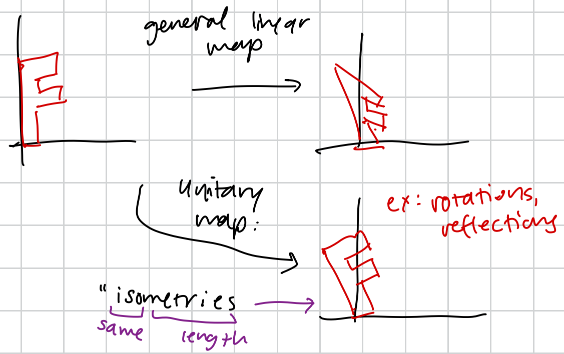
PHILOSOPHY (why we care)

T being unitary, we know:
T is linear and invertible: isomorphism of vector space
($[T]$ square)

intuitively, translation/renaming between two representations of the same underlying space

$\langle Tv, Tv' \rangle = \langle v, v' \rangle$: T preserves inner products

unitary maps $\longleftrightarrow$ "inner product space isomorphisms"



Ex: $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (reflection across $x=y$ line)

Check T is unitary:

$$[T][T]^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

Q: Let $v_1, \dots, v_n$ be columns of $[T]$.
If $T^*T = I$, what does that say about $v_1, \dots, v_n$?

Pf: Let $[T] := \begin{bmatrix} | & | & & | \\ [v_1] & [v_2] & \cdots & [v_n] \\ | & | & & | \end{bmatrix}$

Then
$$T^* := \begin{bmatrix} - [\bar{v}_1]^T - \\ - [\bar{v}_2]^T - \\ \vdots \\ - [\bar{v}_n]^T - \end{bmatrix}$$

So $[T^*][T] = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$

$$[T^*][T] = \begin{bmatrix} \bar{v}_1^T v_1 & \bar{v}_1^T v_2 & \cdots & \bar{v}_1^T v_n \\ \bar{v}_2^T v_1 & \cdots & & \vdots \\ \vdots & & \ddots & \vdots \\ \bar{v}_n^T v_1 & \cdots & & \bar{v}_n^T v_n \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_n, v_n \rangle \\ \vdots & \ddots & \vdots \\ \langle v_1, v_n \rangle & \cdots & \langle v_n, v_n \rangle \end{bmatrix}$$

multiplying each column by itself to get 1 in diagonal.

$\Rightarrow \bar{v}_j^T \cdot v_i \begin{cases} = 1 & \text{when } i=j \\ = 0 & \text{when } i \neq j \end{cases}$

$\Rightarrow \{v_1, \dots, v_n\}$ are orthonormal.

Take-Home Questions:

Q: what about the rows?

Q: Is the converse true?

General thing of

NORMAL OPS

$T \in L(V,V)$ is normal
$TT^* = T^*T$.

ex: self-adjoint, unitary are normal.

Made with Goodnotes

Guiding Question:
can we say things about eigenvalues of normal operators?

Ex: $T = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

If $T^*T = TT^*$,
what does that tell you more specific about a, b, c?

$b = 0$
It is diagonal

<u>Pf</u>:

$[T]^* = \begin{bmatrix} \bar{a} & 0 \\ \bar{b} & \bar{c} \end{bmatrix}$

$TT^* = T^*T$

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} \bar{a} & 0 \\ \bar{b} & \bar{c} \end{bmatrix} = \begin{bmatrix} \bar{a} & 0 \\ \bar{b} & \bar{c} \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$$\begin{bmatrix} a\bar{a} + b\bar{b} & b\bar{c} \\ c\bar{b} & c\bar{c} \end{bmatrix} = \begin{bmatrix} \bar{a}a & \bar{a}b \\ a\bar{b} & \bar{b}b + \bar{c}c \end{bmatrix}$$

$a\bar{a} + b\bar{b} = \bar{a}a$

$b\bar{b} = 0 \Rightarrow b = 0$

Thus, **T is diagonal.**

(general) Conjecture:
Assume $[T] \in \mathbb{F}^{n \times n}$ is upper-triangular
If T is normal, then $[T]$ is diagonal.

Take-home Q: try on 3×3, try to prove in general.

punchline: A4 HW9 says all operators $V \to V$ are triangularizable (in o.n. B)
So if T normal, then it's diagonalizable in orthonormal basis

Day 3 : Lecture 06/04/25

The spectral theorem

Let $(v, \ <*,*>)$ be an n-dim inner product space over $F = C$.

Thm (Spectral Theorem)

If $T \in L(V)$ is normal, then there exists an ONB of V such that $_B[T]_B$ is diagonal ("unitarily diagonalizable").

Claim

This is equivalent to the statement: If $A \in \mathbb{C}^{nxn}$ is normal, then there exists a unitary $U \in \mathbb{C}^{nxn}$ such that $U^{-1}AU$ is diagonal.

Pf

Suppose that there exists a unitary matrix $U \in \mathbb{C}$ such that $U^{-1}AU$ is diagonal.

$_B[T]_B := {}_B[I]_E\,({}_E[T]_E)\,{}_E[I]_B$. Let $b_j := \sum_{i=1}^{n} u_{ij}e_j$ so that $_E[E]_B = ({}_E[b_1] \dots {}_E[b_n])$. Then $_B[T]_B = {}_B[I]_E\,({}_E[T]_E)\,{}_E[I]$, is diagonal. It remains to be shown that $B := \{b_1 \dots b_n\}$ is an ONB.

Indeed, $< b_i, b_j > = < \sum_{i=1}^{n} u_{ij}e_i, \sum_{i'=1}^{n} u_{i'j}e_{i'} >$

$= \sum_{i=1}^{n} u_{ij} < e_i, \sum_{i'=1}^{n} u_{i'j}e_{i'} > = \sum_{i=1}^{n} u_{ij} < e_i, u_{i'j}e_{i'} >$

$= \sum_{i=1}^{n} u_{ij}\overline{u_{ij}} < e_i, e_i > = \sum_{i=1}^{n} (U^{*}U)_{j'j}$

$= I_{j'j} = \{1,\ j = j',\ 0,\ j \neq j'\}.$

It remains to prove the spectral theorem for matrices. By HW 9 A4, every $A \in \mathbb{C}^{nxn}$ is unitarily triangularizable.

If A is also normal, then in fact, the triangularization is a diagonalization.

The singular-value decomposition (SVD)

Let $A \in \mathbb{C}^{mxn}$ and suppose $m \geq n$. Then, $A^* A$ is self-adjoint, so by the spectral theorem, $A^* A = V \Lambda V^{-1}$ for some diagonal $\Lambda \in \mathbb{C}^{nxn}$ =

$$\begin{vmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_n \end{vmatrix}$$

For some unitary $V \in C^{nxn} = [v_1 ... v_n] \in C^{nxn}$.

Then $A_i \in \mathbb{R}$ and $\lambda_i \geq 0$ because $A^* A$ is self-adjoint. $A^* A v_i = \lambda_i v_i$ =>

$< A^* A v_i, v_i > = < \lambda_i v_i, v_i > = \lambda_i < v_i, v_i >$

=> $||Av_i|| = \lambda_i ||v_i||^2$

=> $\lambda_i = \frac{||Av_i||^2}{||v_i||^2} > 0.$

Discussion 6/5

SVD Recap

We are given any matrix $A \in \mathbb{C}^{m\times n}$ where $m \geq n$.

Note: A^*A is self-adjoint so there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of eigenvectors of A^*A by the Spectral Theorem.

Let λ_i be an eigenvalue corresponding to v_i, then $\sigma_i = \sqrt{\lambda_i}$. We refer to these σ_i as singular values.

Goal: find U, Σ, V such that $A = U\Sigma V^*$

$$V = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$$

Note: $\{v_1, \ldots, v_n\}$ is an orthonormal basis, so V is unitary.

We want to construct U such that it is unitary and Σ such that it is diagonal(ish). Specifically, we want to find that

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ & \mathbf{0}_{(m-n)\times n} & & \end{bmatrix}$$

and

$$U^*A = \Sigma V^*$$

in which case we can define u by

$$u_i = \frac{Av_i}{\sigma_i}$$

It remains to show:

$$U = \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix}$$

Other details to consider:

- What if $\sigma_i = 0$?
- Is U actually unitary?
- $A = U\Sigma V^*$

Final Review

Week 1

Key Terms

- Vector spaces
- Groups
- Commutativity
- Associativity
- Identity
- Inverse
- Closure
- Fields
- Set
- Addition
- Multiplication
- Distributivity
- Compatibility
- Subspaces
- Intersection of subspaces
- Sum of subspaces

Key Facts

- Subspaces are vector spaces
- Unique identity and inverse
- Subspace check: closure properties, existence of 0
- Intersections and sums of subspaces are subspaces
- To prove $A = B$, show $A \subseteq B$ and $B \subseteq A$

Aside:
If we have subspaces $V_1, \ldots, V_n$ where each $V_i = \text{span}\{v_i\}$, then

$$W = V_1 + \cdots + V_n \iff W = \text{span}\{v_1, \ldots, v_n\}.$$

Moreover,

$$W = V_1 \oplus V_2 \oplus \cdots \oplus V_n \iff W = \text{span}\{v_1, \ldots, v_n\} \quad \text{and} \quad \{v_1, \ldots, v_n\} \text{ is linearly independent.}$$

Week 2

Key Terms

- Linear Independence
- Basis
- Span
- Dimensions

Key Facts

- Linear dependence lemma
- Steinitz Exchange
- all bases same size
- bases are maximally independent and minimally spanning

Week 3

Key Terms

- Linear Maps
- Kernel
- Image
- Nullity
- Rank
- Injectivity
- Surjectivity
- Bijectivity

Key Facts

- a map is linear if it satisfies additivity and homogeneity, can show both at once using $\alpha T(v) + T(w)$
- $T(0) = 0$ if $T \in L(V, W)$
- Rank-Nullity Theorem
- Bijective $\iff$ Surjective and Injective
- $T \in L(V, W)$ is injective $\iff$ $\ker(T) = \{0\}$

- $T \in L(V, W)$ is surjective $\iff W = \text{im}(T)$
- kernel and image are subspaces

Week 4

Key Terms

- Inverse
- Isomorphism
- Matrices
- Matrix Addition
- Matrix Multiplication
- Column Space
- Column Rank
- Row Space
- Row Rank

Key Facts

- matrix multiplication is a representation of function composition
- column k of matrix product AB is A times column k of b
- columns of matrix product AB are linear combinations of columns of A
- rows of matrix product AB are linear combinations of row of B
- transpose of a matrix is the matrix obtained by interchanging the rows and columns
- Change of Basis: columns of a transformation matrix represent transformed coordinates of basis vectors,

  $${}_W[I]_V = [{}_W[v_1] \dots {}_W[v_n]]$$
  $${}_V[I]_W = [{}_V[w_1] \dots {}_V[w_n]]$$
- If $T \in L(V, W)$ is invertible, then T^{-1} is its inverse. $TT^{-1} = I_W$. $T^{-1}T = I_V$.
- invertibility $\iff$ bijectivity
- injectivity is equivalent to surjectivity if $\dim(V) = \dim(W)$ for $T \in L(V, W)$
- two vector spaces are isomorphic $\iff$ have the same dimension

Week 5

Key Terms

- Determinant
- Polynomials
- Polynomial division

Key Facts

- determinant is multilinear, alternating, and normalized
- determinant of an upper triangular matrix is the product of its diagonal entries
- polynomial division theorem

Math 115A Lecture Notes

June 6, 2025

Matrix of a linear map

Let $T \in L(V)$ be a linear map on a vector space V. Let $B = \{v_1, \dots, v_n\}$ be a basis for V. The matrix of T with respect to the basis B is denoted as $[T]_B$.

$$[T]_B = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

This matrix is constructed by applying T to each basis vector and expressing the result in the same basis: $[T]_B = \begin{pmatrix} [Tv_1]_B & [Tv_2]_B & \dots & [Tv_n]_B \end{pmatrix}$

Triangularization and Diagonalization

If $[T]_B$ is an upper triangular matrix, then the eigenvalues of T are the diagonal entries $a_{11}, \dots, a_{nn}$.

If $[T]_B$ is a diagonal matrix: $[T]_B = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$ then the eigenvalues of T are the diagonal entries $a_{11}, \dots, a_{nn}$, and the corresponding eigenvectors are the basis vectors $v_1, \dots, v_n$.

For a general matrix representation, we have $Tv_j = \sum_{i=1}^{n} a_{ij} v_i$. In the case of a diagonal matrix, this simplifies to $Tv_j = a_{jj} v_j$.

The Spectral Theorem relates to the case where the basis B is also orthonormal.

Topics Discussed

- Eigenvalues of linear maps, diagonalization, and triangularization
- The Spectral Theorem and orthonormal diagonalization
- Determinants using the permutation formula
- Minimal Polynomials
- Direct Sums

Direct Sums

Let V be a vector space, and let $U_1, \ldots, U_k$ be subspaces of V. The sum $W = U_1 + \cdots + U_k$ is a direct sum, denoted $W = U_1 \oplus \cdots \oplus U_k$ or $W = \bigoplus_{i=1}^{k} U_i$, if every vector $w \in W$ can be written uniquely as a sum $w = u_1 + \cdots + u_k$, where each $u_i \in U_i$.

Reminder: The sum of two subspaces is defined as $W = U_1 + U_2 = \{u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2\}$.

A condition for a sum to be direct is that for any $i \neq j$, the intersection of the subspaces is the zero vector: $U_i \cap U_j = \{0\}$.

Example in $\mathbb{R}^3$

Let $V = \mathbb{R}^3$. Consider the subspaces: $U_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ $U_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ (Note: The original notes have some ambiguity here, this is one interpretation).

Let's check if $\mathbb{R}^3 = U_1 \oplus U_2$. A vector in $\mathbb{R}^2$ is written as a sum of vectors from U_1 and U_2.

Another example from the notes: Let $U_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$. Let's define U_2 based on Branden's idea. Pick a vector v not in U_1. Let $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and set $U_2 = \text{span}\{v\}$. To check if $\mathbb{R}^3 = U_1 \oplus U_2$, we can form a matrix with the basis vectors of U_1 and U_2 and check if they are linearly independent. The vectors are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. These are linearly independent, so their span is $\mathbb{R}^3$ and the sum is direct.

The notes also show a calculation for a cross product which results in the vector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.