1 Discussion 1 — Introduction to the Adjoint

HW7 B2

Some motivation for our study of adjoints with the proof of HW7 B2 courtesy of Lorelei.

Let V, W be inner-product spaces.

Let the basis for V^3 be orthonormal. Let $\{b_i\}$ be an orthonormal basis for W.

1. Inner Product in Coordinates

For all $v, v' \in V$,

$$\langle v, v' \rangle = [v]^*[v'].$$

That is, if $[v]$ and $[v']$ are coordinate vectors of v and v' in an orthonormal basis, then their inner product is given by the conjugate transpose of one times the other. In particular,

$$\langle v, v \rangle = [v]^*[v].$$

2. Linear Maps and the Adjoint

Let $T: V \to W$ and $S: W \to V$. We define S to be the adjoint of T if

$$\forall v \in V,\ w \in W,\ \langle Tv, w \rangle = \langle v, Sw \rangle.$$

In this case, we write $T = S^*$ and $S = T^*$.

3. Matrix of a Transformation and its Adjoint

Let $\{b_j\}$ be an orthonormal basis for V, and $\{d_i\}$ be an orthonormal basis for W. Define

$$A := [T]_{\mathcal{D} \leftarrow \mathcal{B}}, \qquad C := [S]_{\mathcal{B} \leftarrow \mathcal{D}}.$$

Then the entries of A are

$$A_{ij} = \langle Tb_j, d_i \rangle,$$

while the entries of C are

$$C_{ji} = \langle Sd_i, b_j \rangle = \langle d_i, Tb_j \rangle = \overline{A_{ij}}.\overline{C_{ji}}$$

Thus $C = A^*$.

$$\overline{C_{i,j}} = A_{i_j}$$

is the "Conjugate Transpose.

4. Matrix Form

Let $Tb_j = \sum_i A_{ij} d_i$. Then the matrix representation of T with respect to the bases $\{b_j\} \to \{d_i\}$ is

$$[T] = [\, Tb_1 \mid Tb_2 \mid \cdots \mid Tb_m \,].$$

5. Properties of the Adjoint Operator

If $T, S \in \mathrm{Hom}(V, W)$ and $\langle Tv, w \rangle = \langle v, Sw \rangle$ for all $v \in V$, $w \in W$, then $T = S^*$ and $S = T^*$.

Is the adjoint unique? Yes; in finite-dimensional spaces, the Riesz Representation Theorem guarantees uniqueness (see Axler 7A for the proof).

Is the adjoint operation linear? Yes:

$$(\alpha T + \beta S)^* = \overline{\alpha}\, T^* + \overline{\beta}\, S^*.$$

Involution Property: $(T^*)^* = T$.

Product Rule: $(TS)^* = S^* T^*$.

Matrix Rule: $(AB)^* = B^* A^*$.

Lecture 1 — Adjoints

Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be finite-dimensional inner-product spaces over a field $\mathbb{F}$.

Definition (Adjoint)

The *adjoint* of $T \in \mathcal{L}(V, W)$ is the map $T^*\colon W \to V$ such that

$$\langle Tv, w\rangle_W = \langle v, T^*w\rangle_V \quad (v \in V,\ w \in W).$$

If B_V and B_W are orthonormal bases of V and W respectively, then

$$[T^*]_{B_W \leftarrow B_V} = \big([T]_{B_V \leftarrow B_W}\big)^*,$$

that is, the conjugate transpose. (Recall that for a unitary matrix A, one has $A^*A = I$.)

Propositions

(a) **Composition.** If $S \in \mathcal{L}(W, U)$ and $T \in \mathcal{L}(V, W)$, then

$$(S \circ T)^* = T^* \circ S^*.$$

Proof. For $u \in U$ and $v \in V$,

$$\langle S(Tv), u\rangle_U = \langle Tv, S^*u\rangle_W = \langle v, T^*S^*u\rangle_V.$$

Hence, $(S \circ T)^* = T^*S^*$.

(b) **Inverse.** If T is invertible, then $(T^{-1})^* = (T^*)^{-1}$.

(c) **Double Adjoint.** $(T^*)^* = T$.

Self-Adjoint Operators

Definition 1 (Self-adjoint)**.** *A linear map $T \in \mathcal{L}(V)$ is* self-adjoint *(also called Hermitian) if $T = T^*$.*

Classroom remarks.

- Nikhil: "For a self-adjoint operator we have $T = T^*$ — the operator is its own adjoint."
- With respect to an orthonormal basis of V, a self-adjoint operator has a Hermitian matrix:

$$[T] = [T]^* \quad \text{(conjugate transpose)}.$$

Eigenvalues and Eigenvectors

Gabriel: "What does self-adjointness tell us about the eigenvalues?"

If $Tv = \lambda v$ with $v \neq 0$, then

$$\langle Tv, v\rangle = \lambda\langle v, v\rangle = \overline{\lambda}\langle v, v\rangle \implies \lambda \in \mathbb{R}.$$

Hence all eigenvalues of a self-adjoint operator are real.

Moreover, if $Tv = \lambda v$ and $Tw = \mu w$ with $\lambda \neq \mu$, then

$$\lambda\langle v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \mu\langle v, w\rangle,$$

so $\langle v, w\rangle = 0$. Thus eigenvectors corresponding to distinct eigenvalues are orthogonal.

Orthogonality Hint

A handy identity for self-adjoint T is

$$\langle Tu, v\rangle + \langle Tv, u\rangle = 0 \quad \Rightarrow \quad \langle Tu, v\rangle = 0.$$

Taking $u = v$ shows that $\langle Tu, u\rangle = 0$ implies $Tu = 0$; hence, if $\langle Tu, v\rangle = 0$ for all $u, v \in V$, then $T = 0$.

Discussion 2 — Introduction to Self-Adjoint Operators and their Properties

Recall (Adjoints)

A linear map $T : V \to V$ is *self-adjoint* if $T = T^*$; i.e. for all $u, v \in V$,

$$\langle Tu, v\rangle = \langle u, Tv\rangle$$

1. **Q: (Kye)** What does this say about $[T]$?

   A: (Brandon): $[T]$ is real and symmetric

   A: (Nikhil): $T = T^* \Rightarrow [T] = \overline{[T]}^T$ (conjugate transpose)

   $$\textbf{Counter: } [T] = \begin{bmatrix} i & i-1 \\ i+1 & i \end{bmatrix}$$

   $[T]$ doesn't necessarily have to be real
   $\Rightarrow$ diagonals are real

2. **Q: (Kye)** If T is self-adjoint, what can we say about $\langle Tv, v\rangle$?

   A: (Gabriel & Kaelan):

   Since T is self-adj., $\langle Tv, v\rangle = \langle v, Tv\rangle$ and by inner product rules,

   $$\langle Tv, v\rangle = \overline{\langle Tv, v\rangle} \Rightarrow \langle Tv, v\rangle \in \mathbb{R}$$

3. **Q:** Do we know anything about the eigenvalues of T? (when T is self-adj.)

Conjecture

If T is self-adj., then its eigenvalues are real.
Proof (Kayla's solution):
Suppose $Tv = \lambda v$

$$\langle Tv, v\rangle = \lambda\langle v, v\rangle = \overline{\lambda}\langle v, v\rangle \Rightarrow \lambda = \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}$$

4. **Q:** If $\langle Tv, v\rangle = 0$ for all $v \in V$ and T is self-adj.?

   Hint: $\langle T(u + iv), u + iv\rangle = 0$

   Proof (Gabriel & friends):

   $$\langle T(u + iv), u + iv\rangle = 0$$

   $$= \langle Tu + iTv, u + iv\rangle = \langle Tu, u + iv\rangle + \langle iTv, u + iv\rangle$$

   $$= \langle Tu, u\rangle + i\langle Tu, v\rangle + i\langle Tv, u\rangle - \langle Tv, v\rangle = 0$$

   Since $\langle Tv, v\rangle = 0 \Rightarrow \|Tv\|^2 = 0 \Rightarrow Tv = 0 \Rightarrow T = 0$

5. **Q:** Suppose T is self-adj. and we know $\ker T \perp \operatorname{Im} T$

   Let $v \in \operatorname{Im}(T) \Rightarrow v = T(u)$

   Then $\langle v, x\rangle = \langle T(u), x\rangle = \langle u, T(x)\rangle$

   If $x \in \ker(T)$, $T(x) = 0 \Rightarrow \langle u, 0\rangle = 0 \Rightarrow \langle v, x\rangle = 0$

   $$\Rightarrow \ker(T) \perp \operatorname{Im}(T)$$

6. **Bonus puzzle:** T is not necessarily self-adjoint
7. **Q:** $T + T^* = T^* + T$

   So $T + T^*$ is self-adjoint even if T isn't necessarily self-adj.
8. **Q:** What about TT^*?

$$(TT^*)^* = T^{**}T^* = TT^* \Rightarrow \text{self-adjoint}$$

9. **Q:** What about T^*T?

Lecture 2 — Self-Adjoint and Normal Operators

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition (Self-Adjoint Operator)

Let $T \in \mathcal{L}(V)$. We say that T is **self-adjoint** if

$$T = T^*.$$

Properties:

- If T is self-adjoint, then TT^* and T^*T are also self-adjoint.
- If $T = T^*$, then for all $v \in V$,
$$\langle Tv, v \rangle = \langle v, Tv \rangle = \overline{\langle Tv, v \rangle} \in \mathbb{R}.$$
Hence, all eigenvalues of T are real.
- If $\langle Tv, v \rangle = 0$ for all $v \in V$, then $T = 0$.

Definition (Normal Operator)

Let $T \in \mathcal{L}(V)$. We say that T is **normal** if

$$TT^* = T^*T.$$

Remarks:

- Every self-adjoint operator is normal.

Proposition 1

T is normal if and only if $\|Tv\| = \|T^*v\|$ for all $v \in V$.

Pf: Suppose T is normal. Then

$$\|Tv\|^2 = \langle Tv, Tv \rangle = \langle T^*Tv, v \rangle,$$
$$\|T^*v\|^2 = \langle T^*v, T^*v \rangle = \langle TT^*v, v \rangle.$$

Since $TT^* = T^*T$, it follows that

$$\|Tv\|^2 = \|T^*v\|^2 \quad \Rightarrow \quad \|Tv\| = \|T^*v\|.$$

Conversely, suppose that for all $v \in V$, $\|Tv\| = \|T^*v\|$. Then

$$\|Tv\|^2 = \|T^*v\|^2 \Rightarrow \langle T^*Tv, v \rangle = \langle TT^*v, v \rangle \Rightarrow \langle (T^*T - TT^*)v, v \rangle = 0.$$

Since this holds for all $v \in V$, it follows that $T^*T = TT^*$, so T is normal.

Proposition 2

If T is normal and $Tv = \lambda v$, then

$$T^*v = \overline{\lambda} v.$$

Pf: Suppose $Tv = \lambda v$ and T is normal. Then

$$\|Tv\|^2 = |\lambda|^2 \|v\|^2, \quad \|T^*v\|^2 = \|Tv\|^2 = |\lambda|^2 \|v\|^2.$$

Also,

$$TT^*v = T^*Tv = T^*(\lambda v) = \lambda T^*v,$$

so T^*v is also an eigenvector with eigenvalue $\overline{\lambda}$. Therefore,

$$T^*v = \overline{\lambda} v.$$

The proofs of proposition 1 and 2 above are courtesy of Gabriel and Ari.

Complex Case: Real and Imaginary Parts of an Operator

Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then

$$T = \frac{T + T^*}{2} + i \cdot \frac{T - T^*}{2i} = \mathrm{Re}(T) + i\,\mathrm{Im}(T),$$

where:

$$\mathrm{Re}(T) = \frac{T + T^*}{2}, \quad \mathrm{Im}(T) = \frac{T - T^*}{2i}.$$

Properties:

- $\mathrm{Re}(T)^* = \mathrm{Re}(T)$
- $\mathrm{Im}(T)^* = \mathrm{Im}(T)$
- If T is normal, then

$$\mathrm{Re}(T)\,\mathrm{Im}(T) = \mathrm{Im}(T)\,\mathrm{Re}(T).$$