Math 115A Week 8 Notes

May 24, 2025

1 Monday

05/19/2025

Definition 1.1. (complex numbers)
$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$ where i is a special number with property $i^2 = -1$

Operations on complex numbers:

1. $(a + bi) + (c + di) = a + c + (b + d)i$
2. $(a + bi)(c + di) = ac - bd + bci + adi = (ac - bd) + (ad + bc)$
3. $\overline{a + bi} = a - bi$

Definition 1.2. (normed vector space)
A normed vector space is a vector space over $\mathbb{C}$ or $\mathbb{R}$ with a norm function $\|\cdot\| : V \to \mathbb{R}$ satisfying these properties:

1. $\forall v, \|v\| = 0$
2. $\forall v \in V$ if $\|v\| = 0$, then $v = 0$
3. $\forall v \in V$, $\forall c \in \mathbb{F}$ $\|cv\| = |c| \|v\|$
4. $\forall u, v \in V$, $\|u + v\| \leq \|u\| + \|v\|$

Definition 1.3. (inner product space)
is a vector space V, with a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$. satisfying properties:

1. linear in first argument

   (a) $\forall u, v, w \in V, \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

   (b) $\forall v, w \in V, \forall c \in \mathbb{F} < cv, w >= c < v, w >$

2. $\forall v, w \in V, < v, w >= \overline{< w, v >}$ "conjugate symmetry"
   Notice: if $z \in \mathbb{R}$, $z = \overline{z}$

3. $\forall v \in V$, $< v, v > \in R \geq 0$, and [if $< v, v >= 0$ then $v = 0$]

Question:(Avik)
Why did Kye not say $< \cdot, \cdot >$ is linear in the second argument?

Question: (Kye)
Given a vector space $\mathbb{C}^2$. Come up with

1. A valid norm function?

2. inner product?

Answers:
Question (Avik), Answer():

$$\begin{aligned} < u, av + bw > &= \overline{< av + bw, u >} \\ &= \overline{a < v, u > + b < w, u >} \\ &= \overline{a < v, u >} + \overline{b < w, u >} \\ &= a < u, v > + b < u, w > \end{aligned}$$

Lemma 1.1. *For* $z_1, z_2 \in \mathbb{C}$, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Question(Kye):
Answer(Kyle): For $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, the norm is $\sqrt{z_1\overline{z_1} + z_2\overline{z_2}}$
Answer(Jasmine): For $\begin{pmatrix} a + bi \\ c + di \end{pmatrix}$, the norm is $|a| + |b| + |c| + |d|$

2 Tuesday

5/20/2025

For $x, y \in \mathbb{C}$:

1. $\overline{x+y} = \overline{x} + \overline{y}$

   Proof. let $x = a + bi$, $y = c + di$

$$\begin{aligned}\overline{(a+c+(b+d)i)} &= a + c - (b+d)i \\ &= a - bi + c - di \\ &= \overline{x} + \overline{y}\end{aligned}$$

$\square$

2. $\overline{xy} = \overline{x} \cdot \overline{y}$

   Proof. let $x = a + bi$, $y = c + di$

$$\begin{aligned}\overline{(a+bi)(c+di)} &= \overline{(a+bi)(c+di)} \\ &= ac - bd - (ad+bc)i \\ &= ac - adi - bci - bd \\ &= (a-bi)(c-di) \\ &= \overline{x} \cdot \overline{y}\end{aligned}$$

$\square$

Q: is $\langle \cdot, \cdot \rangle$ linear in second arg?

1. linear in addition

$$\begin{aligned}\langle u, v+w \rangle = \overline{\langle v+w, u \rangle} &= \overline{\langle v, u \rangle + \langle w, u \rangle} \\ &= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\ &= \langle u, v \rangle + \langle u, w \rangle\end{aligned}$$

2. Linear in scalar multiplication

$$\begin{aligned}\langle u, cv \rangle = \overline{\langle cv, u \rangle} &= \overline{c\langle v, u \rangle} \\ &= \overline{c}\langle u, v \rangle\end{aligned}$$

Q: If I have an inner product, do I also have a norm?
Define:

$$\|v\| = \sqrt{\langle v, v\rangle}$$

1. Non-negative
   As $\langle v, v\rangle$ is non-negative, $\sqrt{\langle v, v\rangle}$ must also be non-negative.

2. Positive Definition

$$\begin{aligned}\|cv\| &= \sqrt{\langle cv, cv\rangle} \\ &= \sqrt{c\langle v, cv\rangle} \\ &= \sqrt{c\overline{\langle cv, v\rangle}} \\ &= \sqrt{c\bar{c}\langle v, v\rangle} \\ &= |c|\sqrt{\langle v, v\rangle} \\ &= |c|\,\|v\|\end{aligned}$$

3. Triangle Inequality
   Proved later on

4. Absolute Homogeneity

$$\begin{aligned}\|v\| = 0 &\implies v = 0 \\ \sqrt{\langle v, v\rangle} &= 0 \\ \langle v, v\rangle &= 0 \\ v &= 0\end{aligned}$$

Side Quests: If we're given u,v satisfying $\langle u, v\rangle = 0$ what is $\|u + v\|$?

$$\begin{aligned}\|u + v\| &= \sqrt{\langle u + v, u + v\rangle} \\ &= \sqrt{\langle u, u + v\rangle + \langle v, u + v\rangle} \\ &= \sqrt{\langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle} \\ &= \sqrt{\langle u, u\rangle + \langle v, v\rangle} \\ &= \sqrt{\|u\|^2 + \|v\|^2}\end{aligned}$$

Thus $\|u + v\|^2 = \|u\| + \|v\|$, which is the Pythagorean Theorem.

Definition 2.1. $u, v \in V$ are orthogonal if $\langle u, v\rangle = 0$

3 Wednesday

05/22/2025

Let $V, \langle \cdot, \cdot \rangle)$ be an inner prod space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

Orthogonal Projections
Let $\|v\| = \sqrt{\langle v, v \rangle}$. Then $\|v + w\|^2 = \|v\|^2 + 2Re\langle v, w \rangle + \|w\|^2$

Prop: (Pythagorean Theorem)
If $\langle v, w \rangle = 0$, then $\|v + w\|^2 = \|v\|^2 + \|w\|^2$

Thm:
$(V, \|\cdot\|)$ is an normed space (i.e. $\|\cdot\|$ is indeed a norm)

It remains to show that $\|v + w\| \leq \|v\| + \|w\|($ for all $v, w \in V)$, which would follow from $Re\langle v, w \rangle \leq \|v\| \cdot \|w\|$

Q: Given $v, w \in V$ w/ $w \neq 0$, find $\alpha \in \mathbb{F}$ st $v - \alpha w \perp w$.

A:

$$\begin{aligned} v - \alpha w \perp w &\iff \langle v - \alpha w, w \rangle = 0 \\ &\iff \langle v, w \rangle - \alpha \langle w, w \rangle = 0 \\ &\iff \alpha = \frac{\langle v, w \rangle}{\langle w, w \rangle} = \frac{\langle v, w \rangle}{\|w\|^2} \end{aligned}$$

Def: The (Orthogonal) Projection of v onto w is $proj_w(v) := \frac{\langle v, w \rangle}{\langle w, w \rangle} w$

Notice that:

$$\begin{aligned} \|proj_w(v)\| \leq \|v\| &\iff \left\| \frac{\langle v, w \rangle}{\|w\|^2} w \right\| \leq v \text{ (*)} \\ &\iff \frac{|\langle v, w \rangle|}{\|w\|^2} \|w\| \leq \|v\| \\ &\iff |\langle v, w \rangle| \leq \|v\| \, \|w\| \end{aligned}$$

This inequality is called the **Cauchy-Schawrz Inequality**
Note, if $|\langle v, w \rangle| \leq \|v\| \, \|w\|$ then $Re\langle v, w \rangle \leq \|v\| \, \|w\|$

and (*) holds b/c:
$\|v\|^2 = \|\alpha w\|^2 + \|v - \alpha w\|^2$ by Pythagorean thm ($v - \alpha w \perp \alpha w$ by def of α)

Then $\|v\|^2 = \|\alpha w\|^2 + \|v - \alpha w\|^2 \geq \|proj_w(v)\|^2$

Prop:
$\|v - proj_w(v)\| \leq \|v - w'\|$ for all $w' \in span\{w\}$ (ie projection minimizes the distance to v within $span\{w\}$), with equality iff $w' = proj_w(v)$.

Proof:

$$\begin{aligned}
&\|v - \beta w\|^2 = \|v - \alpha w\|^2 + \|\beta w - \alpha w\|^2 \\
&\|v - \beta w\|^2 \geq \|v - \alpha w\|^2 \\
&\text{if } \|v - \beta w\|^2 = \|v - \alpha w\|^2 \implies \alpha = \beta
\end{aligned}$$

Is $v - \alpha w \perp \beta w - \alpha w$?
$\langle v - \alpha w, (\beta - \alpha)w \rangle = \overline{\beta - \alpha} \langle v - \alpha w, w \rangle = 0$

Generalize: Given $v \in V$ and a subspace W w/ basis $\{w_1, \cdots, w_m\}$, find $w \in W$ st $v - w \perp W$, ie, $v \perp w'$ for all $w' \in W$.

By linearity, it suffices to find scalars $\alpha_q, \cdots, \alpha_m \in \mathbb{F}$ st for each j, $\langle v - \sum_{i=1}^{m} \alpha_i w_i, w_j \rangle = 0$

4 Thursday

05/22/2025

Fix $v, w \in V$, $w \neq 0$, then $proj_w(v) = \frac{\langle v,w \rangle}{\langle w,w \rangle}$
Useful Properties:

1. minimizes distance to v among all elements $m \in span\{w\}$
2. $v - proj_w(v) \perp w$
3. $\|v - proj_w(v)\| \leq \|v - w'\|$

Q: Given $\{w_1 w_2\}$ basis of subspace $W = span\{w_1, w_2\}$.
Given V, come up with a formula for $proj_W(v)$ so that.

1. $\forall x \in W$, $v - proj_W(v) \perp x$
2. $\forall x \in W, \|v - proj_W(v)\| \leq \|v - x\|$

Assume that w_1, w_2 are orthonormal:

1. $w_1 \perp w_2$
2. $\|w_1\| = \|w_2\| = 1$

Proposal:
$proj_W(v) = \frac{\langle v,w_1\rangle}{\langle w_1,w_1\rangle} w_1 + \frac{\langle v,w_2\rangle}{\langle w_2,w_2\rangle} w_2$

If $w_1 \cdots w_n$ orthonormal, define:
$proj_W(v) := \sum_{i=1}^{n} \frac{\langle v,w_i\rangle}{\langle w_i,w_i\rangle} w_i$

Prove:

1. $\forall w \in W$, $\langle v - proj_W(v), w \rangle = 0$
2. $\forall w' \in W$, $\|v - projW(v)\| \leq \|v - w'\|$

Given basis $B = \{b_1, \cdots, b_n\}$
We want to construct an orthonormal set using B.

Define:

$$
\begin{aligned}
b'_1 &= b_1 \\
b'_2 &= b_2 - proj_{b'_1} b_2 \\
b'_3 &= b_3 - proj_{b'_1} b_3 - proj_{b'_2} b_3 \\
&\vdots \\
b'_k &= b_k - \sum_{i=1}^{k-1} proj_{b'_i} b_k
\end{aligned}
$$

Finally, define $\forall i = 1 \cdots n$
$\hat{b_i} = \frac{b'_1}{\|b'_i\|}$

OMG GRAHAM SHMIDT
If $\hat{b}_i \cdots \hat{b}_n$ orthonormal then linearly independent and spanning?
Why does this not fail, how come $b'_i \neq 0$
If $\hat{B} = \{\hat{b}_1, \cdots, \hat{b}_n\}$ is basis? what is $\|x\|$ in terms of $[x]_{\hat{B}}$

4.1 Friday

05/25/2025

$\forall w \in W$, $v - proj_w \perp w'$

Proof. (Jamie)
Consider some $\langle v - proj_w v, w' \rangle$, by linearity in the first term

$$\begin{aligned}
\langle v - proj_w v, w' \rangle &= \langle v, \sum_{i=1}^{m} \alpha_i w_I \rangle - \langle \sum_{i=1}^{m} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i, \sum_{j=1}^{m} \alpha_j w_j \rangle \\
&= \sum_{i=1}^{m} \overline{\alpha_i} \langle v, w_i \rangle - \sum_{i=1}^{m} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \sum_{j=1}^{m} \alpha_j \langle w_i, w_j \rangle \\
&\text{As W is orthonormal all inner product of } i \neq j \text{ are 0} \\
&= \sum_{i=1}^{m} \overline{\alpha_i} \langle v, w_i \rangle - \sum_{i=1}^{m} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \alpha_i \langle w_i, w_i \rangle \\
&= \sum_{i=1}^{m} \overline{\alpha_i} \langle v, w_i \rangle - \sum_{i=1}^{m} \alpha_i \langle v, w_i \rangle = 0
\end{aligned}$$

□

$\|v - proj_w v\| \leq \|v - w'\|$

Proof. Observe, $proj_w v - w' \in W$
Thus, $v - proj_w v \perp proj_w v - w'$
By Pythagorean Theorem:

$$\|v - w'\|^2 = \|v - proj_w v\|^2 + \|proj_w v - w'\|^2 \geq \|v - proj_w v\|$$

□

Definition 4.1. (Orthogonal Complement)
For a subset $W \subseteq V$, the Orthogonal Complement is:

$$W^{\perp} := \{v \in V : w \in W, \langle v, w \rangle = 0\}$$

Prop: Let $W \subseteq V$, then $W^{\perp}$ is a subspace of V.

Proof. (Avik)
WTS: $0 \in W^{\perp}$, and that $W^{\perp}$ is closed under addition and scalar multiplication.

1. $\langle 0, w \rangle = 0$
2. $\forall v, w \in V$ st $\langle v, w \rangle = 0$, For $\alpha \in \mathbb{F}$, $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle = 0$
3. $\forall v_1, v_2, w$ st $\langle v_1, w \rangle = 0$ and $\langle v_2, w \rangle - 0$:
   $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle = 0$

□

If W is a finite dimensional subspace of V and $v \in V$, we showed that $w := proj_w(v)$ is the unique vector satisfying:

1. $w \in W$
2. $v - w \in W^{\perp}$

Prop: Let W be a finite dimensional subspace of V and $P_w := proj_w$, then $P_w \in L(W)$

Proof. (Alan)

1. $\forall v \in V, P_w(v) \in V$
   As all $P_w(v) \in W$ and $W \subseteq V$, then $P_w(v) \in V$
2. $P_w(\alpha v_1 + v_2) = \alpha P_w(v_1) + P_w(v_2)$
   $\alpha v_1 + v_2 + \alpha P_w(v_1) + P_w(v_2) \in W$
   As for any vector v there exists a unique w such that $v - w \in W^{\perp}$, thus $P_w(v_1) + P_w(v_2) = P(\alpha v_1 + v_2)$

□