Math 115A: Week 7 Notes

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1 Eigenvalues and Eigenvectors Continued

MONDAY LECTURE

Let V be an n-dimensional vector space over a field $\mathbb{F}$-vector space, and $T \in L(V)$.

Remark 1.1. We sometimes call linear maps "linear operators".

Definition 1.2. If λ is an eigenvalue of T, its

Geometric Multiplicity $\gamma_T(\lambda)$ is

$$\dim(\underbrace{\ker(\lambda I - T)}_{\text{Eigenspace} E_T(\lambda)}) = \mathrm{null}(\lambda I - T).$$

$$1 \leq \gamma_T(\lambda) \leq n$$

Algebraic Multiplicity $\alpha_T(\lambda)$ is

the multiplicity of λ as a root of

$$\underbrace{\chi_T(\lambda) \coloneqq \det(\lambda I - T)}_{\text{Characteristic Polynomial}}$$

$$1 \leq \alpha_T(\lambda) \leq n$$

Proposition 1.3. *$\gamma_T(\lambda) \leq \alpha_T(\lambda)$: the geometric multiplicity of an eigenvalue λ is at most the algebraic multiplicity of λ*

Suppose λ is an eigenvalue of T and $\gamma_T(\lambda) =$ m. Then, there exist $v_1, \ldots, v_m$ such that $\{v_1, \ldots, v_m\}$ is a basis of $E_T(\lambda)$. Extend this to a basis $B \coloneqq \{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ of V.

Question 1.4. Questions to think about

1. What is/what form does ${}_B[T]_B$ have?
2. What about the characteristic polynomial of this matrix?

Example 1.5.

$$T(x_1, x_2) = (3x_1 - x_2,\ x_1 + x_2) \longrightarrow \begin{array}{r l} \text{Eigenvalues:} & \lambda = 2, \alpha_T(2) = 2 \\ & m = \gamma_T(2) = 1 \end{array}$$

1.

$$\left[\begin{array}{cccccc|cccccc}
\lambda & 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\
0 & \lambda & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\
0 & 0 & \ddots & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\
0 & 0 & 0 & \ddots & 0 & 0 & ? & ? & ? & ? & ? & ? \\
0 & 0 & 0 & 0 & \lambda & 0 & ? & ? & ? & ? & ? & ? \\
0 & 0 & 0 & 0 & 0 & \lambda & ? & ? & ? & ? & ? & ? \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\
0 & 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\
0 & 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\
0 & 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\
0 & 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\
0 & 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ?
\end{array}\right]$$

$${}_B[Tv_1]_B \quad \cdots \; {}_B[Tv_m]_B \; {}_B[Tv_{m+1}]_B \cdots \; {}_B[Tv_n]_B$$

Back Table: Try different extensions:

$$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\} \quad \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} \quad \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}\right\} \quad \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$$

In each trial,

$${}_B[T]_B = \begin{bmatrix}2 & ?\\0 & 2\end{bmatrix}$$

Gary Observed:By the way, $\det {}_B[T]_B = [T]_{original\ basis}$

2.

$$\det(xI - {}_B[T]_B) = (x-\lambda)^m \quad \cdot \quad ?$$

$x = \lambda$ is necessarily a solution.

*multiplicity of m

2 Eigenvalues and Eigenvectors Continued

TUESDAY DISCUSSION

V is finite-dimensional. Take λ eigenvlaue o fT, v_1, $\cdots$, v_m basis of ker(λI-T). Extend to basis B=v_1, $\cdots$, v_m, v_{m+1}, $\cdots$, v_m of V.

Question 2.1. If we know ${}_B[T]_B$, then what can we say about $\chi_T(x)$?

Claim 2.2. (x-λ) appears at least m times in $\chi_T(x)$.

——————— Observe that:

$$\left[\begin{array}{cccc:cccc} x-\lambda & 0 & 0 & 0 & ? & ? & ? & ? \\ 0 & \ddots & 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & \ddots & 0 & ? & ? & ? & ? \\ 0 & 0 & 0 & x-\lambda & ? & ? & ? & ? \\ \hdashline 0 & 0 & 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? & ? & ? \end{array}\right]$$

Evaluate $\det(xI-T)$ using the permutation expansion:

ways to choose
distinct rows
from each column

$$\sum_{\sigma} \operatorname{sgn}(\sigma)\, a_{\sigma(1),1} \cdot a_{\sigma(2),2} \cdots a_{\sigma(n),n}$$

———————

For each permutation σ, if $\sigma(1) \neq 1$, then the corresponding term sgn(σ) $a_{\sigma(1),1} \cdots a_{\sigma(n),n} = 0$ because $a_{\sigma(1),1} = 0$. So, only the terms with $\sigma(1) = 1$ survive. In fact, by the same reasoning, all terms with $\sigma(2) \neq 2 \cdots \sigma(n) \neq n$ vanishes. So, we know all surviving terms have $\sigma(1) = 1 \cdots \sigma(m) = m$. So, each of these terms shares the factors $a_{11}\ a_{22}\ \cdots\ a_{mm} = (x-\lambda)^m$ We just showed $(x-\lambda)^m$ is a factor in each term of $det(\lambda I - T) = \Sigma$ .... So $(x-\lambda)^m$ is a factor of $det(\lambda I - T)$.
Conclusion:

$$\underbrace{\alpha_T(\lambda)}_{\text{algebraic multiplicity}} \geq m = \underbrace{\gamma_T(w)}_{\text{geometric multiplicity}}$$

Wednesday: Induction / Quiz review

Let V be a vector space over $\mathbb{F}$: $T \in L(V)$

claim: for any n and for any $v_1, \dots, v_n$, if $v_1, \dots, v_n$ are eigenvectors of T with distinct eigenvalues then $v_1, \dots, v_n$ are linearly independent.

→ This proof for $n=2$ was covered yesterday.

→ Today, we did a general proof by induction using $n=2$ as the base case:

using base case from Tuesday:

Proof by Tam:

Inductive hypothesis: Suppose $\sum_{i=1}^{k} \alpha_i v_i = 0$ iff. all $\alpha_i = 0$

where $v_1, \dots, v_n$ are eigenvectors for distinct eigenvalues.

Now suppose $\alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} = 0$ ①

$T(\alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1}) = \alpha_1 T v_1 + \dots + \alpha_k T v_k + \alpha_{k+1} T v_{k+1}$
$= \alpha_1 \lambda_1 v_1 + \dots + \alpha_k \lambda_k v_k + \alpha_{k+1} \lambda_{k+1} v_{k+1} = 0$ ②

Now take ① $\times$ λ_{k+1}:

$\alpha_1 \lambda_{k+1} v_1 + \dots + \alpha_k \lambda_{k+1} v_k + \alpha_{k+1} \lambda_{k+1} v_{k+1} = 0$

Subtract ① from ②:

$$\alpha_1 v_1 (\lambda_1 - \lambda_{k+1}) + \dots + \alpha_k v_k (\lambda_k - \lambda_{k+1}) + \alpha_i (\lambda_i - \lambda_{k+1}) = 0$$

By inductive hypothesis, because $v_1, \dots, v_k$ are L.I. and λ_i and λ_{k+1} are distinct and v_i is an eigenvector,

$$\alpha_i v_i (\lambda_i - \lambda_{k+1}) = 0$$

α_i must equal zero $\rightarrow \alpha_1, \dots, \alpha_k = 0$.

Insert back into ①:

$$\alpha_{k+1} v_{k+1} = 0$$

Since $v_{k+1} \neq 0$, $\alpha_{k+1} = 0$.

Thus, $v_1, \dots, v_{k+1}$ is linearly independent.

$\square$

3 Eigenvalues and Eigenvectors Continued

THURSDAY DISCUSSION

Diagonalization
Let V be an n-dimensional vector space over a field $\mathbb{F}$-vector space, and $T \in L(V)$.

Definition 3.1. T is DIAGONALIZABLE if there exists a basis B of V such that ${}_B[T]_B$ is diagonal.

Let $a_{ij} \in \mathbb{F}$. Then the diagonal matrix is given by:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

Theorem 3.2. *If λ_1, ..., λ_m are the distinct eigenvalues of T (these numbers are different from each other), then the following are equivalent:*

(a) T is diagonalizable
(b)V has a basis fo eigenvectors of T

$$(c) \qquad V = \bigoplus_{i=1}^{m} E_T(\lambda_i)$$

Direct Sum: *For all $v \in V$, there exists unique $v_i \in E_T(\lambda_i)$ such that*

$$v = \sum_{i=1}^{m} v_i$$

$$(d) \qquad n = \sum_{i=1}^{m} \gamma_T(\lambda_i)$$

(e) $\quad \chi_T$ has n zeroes (counted with multiplicity), and $\gamma_T(\lambda_i) = \alpha_T(\lambda_i)$ for each i.

Counted Multiplicity:

$$\sum_{i=1}^{m} \alpha_i(\lambda_i) = n$$

Thursday

Diagonalization

Let V be a n-dim. $\mathbb{F}$-vector space.
and $T \in L(V)$.

Def T is diagonalizable if there exists a basis B of V s.t. ${}_B[T]_B$ is diagonal. $\rightarrow \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$

$a_{ij} \in \mathbb{F}$

Let V be a n-dim. $\mathbb{F}$-vector space.
and $T \in L(V)$.

these numbers are diff. from each other.

Theorem If $\lambda_1, \dots, \lambda_m$ are the distinct eigenvalues of T, then the following are equivalent:

if any 1 is true, all are true

if any 1 is false, all are false

(a) T is diagonalizable

(b) V has a basis of eigenvectors of T

(c) $V = \bigoplus_{i=1}^{m} E_T(\lambda_i)$

direct sum:
for all $v \in V$, there exists unique $v \in E(\lambda_i)$ s.t. $v = \sum_{i=1}^{m} v_i$

(d) $n = \sum_{i=1}^{m} \gamma_T(\lambda_i)$

$\sum_{i=1}^{m} \alpha_i(\lambda_i) = n$

(e) χ_T has n zeroes (with counted multiplicity) and $\gamma_T(\lambda_i) = \alpha_i(\lambda_i)$ for each i

We will prove:

$$(a) \rightleftarrows (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (b)$$

Kye's Proof:

Claim (b) $\Rightarrow$ (c)

If V has a basis of eigenvectors of T, then $V = \bigoplus_{i=1}^{m} E_T(\lambda_i)$

Proof: Assume V has a basis of eigenvectors of T, Call it $B = \{v_{1,1}, v_{1,2}, \dots, v_{1,k}, v_{2,1}, \dots, v_{2,k}, \dots, v_{m,1}, \dots, v_{m,k_m}\}$

grouped by eigenvalues:
$v_{i,j}$ is the j-th eigenvector for eigenvalue λ_i. $\quad k_i = \#$ of basis vectors belonging to eigenvalue λ_i

Need to show:
① For all $v \in V$, $\exists v_i \in E_T(\lambda_i)$ s.t. $v = v_1 + \dots + v_m$

② Unique: if $u_i \in E_T(\lambda_i)$ also satisfies $v = u_1 + \dots + u_m$, then $u_1 = v_1 \dots v_m = u_m$.

Pf ①:

Let $v \in V$ be given.

B is a basis, so it spans.

so let $c_{i,j}$ be coefficients satisfying

$$v = \sum_{i=1}^{m} \sum_{j=1}^{k_i} c_{i,j} v_{i,j}$$

Notice for each i, each $v_{i,j}$ has eigenvalue λ_i, meaning by closure of eigenspaces.

$$\sum_{j=1}^{k_i} c_{i,j} \underbrace{v_{i,j}}_{\in E_T(\lambda_i)} \in E_T(\lambda_i)$$

So let $v_i = \sum_{j=1}^{k_i} c_{i,j} v_{i,j}$

And we have $v = \sum_{i=1}^{m} v_i$.

Pf ②: $v_i \in E_T(\lambda_i)$

Uniqueness: Know $v = v_1 + \dots + v_m$

Suppose $v = u_1 + \dots + u_m$ for some $u_i \in E_T(\lambda_i)$, $i = 1, \dots, m$.

Then: $0 = \underbrace{(u_1 - v_1)}_{\in E_T(\lambda_1)} + \dots + \underbrace{(u_m - v_m)}_{\in E_T(\lambda_m)}$ ☹

Know: $\lambda_1, \dots, \lambda_m$ distinct

We proved on quiz/yesterday that eigenvectors w/ distinct eigenvalues are linearly independent.

Each $u_i - v_i$ is either an eigenvector or 0.

But if $\exists i$ s.t. $u_i - v_i \neq 0$ then it is an eigenvector, so ☹ shows linear dependence of the nonzero terms, contradiction.

So, for all i, $u_i - v_i = 0$

i.e. $u_i = v_i$

Friday

Brandon's Proof

(b) $\Rightarrow$ (a)

V has a basis of eigenvectors of $T \Rightarrow T$ is diagonalizable.

V has a basis $\{v_1, \dots, v_m\}$ B is a basis: $Tv_i = \mu_i v_i$

$$_B[T]_B = [_B Tv_1 \cdots {}_B Tv_m] = [_B \lambda_1 v_1 \cdots {}_B \lambda_n v_n] = \begin{bmatrix} \mu_i & 0 & 0 & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \\ 0 & & & \mu_n \end{bmatrix}$$

So T is diagonal.

Kyle + Eric:

(c) $\Rightarrow$ (d)

$E_T(\lambda_i)$

$B_i = \{v_{i1}, \dots, v_{im}\}$ $\dim(E_T) = m$ $\dim(E_i) = \gamma_T(\lambda_i)$ ... (Proof in progress)

Kye's Support:

Ericonjecture: If $\mathcal{U} = V \oplus W$

$$\dim \mathcal{U} = \dim V + \dim W$$

$\forall u \in U, \exists v \in V, w \in W$:

so that,

$$u = v + w$$

where,

v, w are unique

$v + w = \alpha_1 v_1 + \dots + \alpha_n v_n + \beta_1 w_1 + \dots + \beta_m w_m = 0$

Idea: try to find a basis of $\mathcal{U}$.

$B_V = \{v_1, \dots, v_n\}$

$B_W = \{w_1, \dots, w_m\}$

$V = \alpha_1 v_1 + \dots + \alpha_n v_n$

$W = \beta_1 w_1 + \dots + \beta_n w_b$

Kyle: if $\mathcal{U} = V \oplus W$

then $V \cap W = \emptyset$

Avik: wrong! $V \cap W$ contains 0.