Math 115A Week 6 Scribe Notes

By Jessica Guan and Briana Waters

Monday, May 5 (Lecture 1)

Question:

S((x1, x2, x3, …)) = (0, x1, x2, x3, …)

Prove that S is injective but not surjective.

Kye's Proof: Notice that if two sequences (x1, x2, x3, …) and (y1, y2, y3, …) are equal, then each of the entries are also equal: x1 = y1, x2 = y2, etc. So, S((x1, x2, x3, …)) = (0, x1, x2, x3, …) and S((y1, y2, y3, …)) = (0, y1, y2, y3, …), but since each yi is equal to the xi's and the only change is that we added 0 to the start of both sequences and 0 = 0, the sequences are the same. Observe that S((x1, x2, x3, …)) = (0, x1, x2, x3, …), so the output contains all the inputs, and adding 0 to a list does not remove anything, so it is surjective.

*** This is a bad proof. ***

Example Exercise:

Let V be a vector space and suppose that v1, …, vm ∈ V are linearly independent and w ∈ V. Prove that dim(span{v1 + w, …, vm + w}) >= m - 1.

Kye's Idea:

S = span{v1 + w, …, vm + w}

To show dim S = m - 1, hopefully it suffices to find a linearly independent set of size m - 1 ∈ S.

We know that v1, …, vm are linearly independent.

We will try the following:

Define u1 = (v1 + w) - (vm + w) = v1 - vm, u2 = (v2 + w) - (vm + w) = v2 - vm, …, um-1 = (vm-1 + w) - (vm + w) = vm-1 - vm

Conjecture: u1, …, um-1 are linearly independent

Suppose c1, …, cm-1 ∈ F satisfy:

$0 = c_1 * u_1 + \ldots + c_{m-1} * u_{m-1}$

$0 = c_1 * (v_1 - v_m) + \ldots + c_{m-1} * (v_{m-1} - v_m)$

$0 = c_1 * v_1 + c_2 * v_2 + \ldots + c_{m-1} * v_{m-1} - (c_1 + \ldots c_{m-1}) * v_m$

By independence of $\{v\}$, $c_1 = \ldots = c_{m-1} = 0$

Check our original idea: Does knowing that $u_1, \ldots, u_{m-1}$ are linearly independent mean dim S >= m - 1?

Let B be a basis of S. By spanning, $u_1, \ldots, u_{m-1}$ independent implies dim S = |B| >= m - 1 by Steinitz exchange lemma.

<u>Kye's Proof:</u>

To simplify notation, define:

$S = \text{span}\{v_1 + w, \ldots, v_m + w\}$

$u_i = v_i - v_m$ for $i = 1, \ldots, m-1$

$U = \{u_1, \ldots, u_{m-1}\} \subset S$

Lemma: U is linearly independent

Proof: Suppose $c_1, \ldots, c_{m-1} \in F$ satisfy $c_1 * u_1 + \ldots + c_{m-1} * u_{m-1} = 0$.

WTS $c_1, \ldots, c_{m-1} = 0$

Observe $0 = c_1 * u_1 + \ldots + c_{m-1} * u_{m-1}$ (by algebra)

$= c_1 * (v_1 - v_m) + \ldots + c_{m-1} * (v_{m-1} - v_m)$

By linear independence of $v_1, \ldots, v_m$, we conclude $c_1, c_2, \ldots, c_{m-1}, (c_1 + c_2 + \ldots + c_{m-1}) = 0$

□

Claim: dim S >= m - 1

Proof: Let B be a basis of S.

We know that dim S = |B|

Note that:

- B is spanning (by definition of basis)
- U is linearly independent (by lemma)

By Steinitz exchange lemma, dim S = |B| >= |U| = m - 1

□

Tuesday, May 6 (Discussion 1)

Consider the following scenario

Oscar's friends: Kyle

Kyle's friends: Oscar, Heather

Isabel's friends: Oscar, Kyle, Isabel, Heather

Heather's friends: Oscar, Kyle, Isabel, Heather

*** On day 0, Kyle receives a scary email in his inbox. On day 1, Kyle wakes up and sees this email. He then forwards it to everyone who is a friend of his before deleting it from his inbox.

This process continues each day with each person, so the number of emails that each person has on the first 3 days is as follows (if a person is friends with themselves, they forward the email to themselves):

	Day 1	Day 2	Day 3
Oscar	1	1	4
Kyle	0	2	3
Heather	1	1	4
Isabel	0	1	2

On day 3, the approximate proportion of the total emails that each person possesses is as follows:

Oscar: ~30%

Kyle: ~23%

Heather: ~30%

Isabel: ~15%

(Note: this is approximate, so it does not add up to 100%.)

Question: Does this distribution of emails approach anything? (If so, what?)

Define the starting condition as follows:

Day "0"

$x_0 =$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Day "1"

$x_1 =$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Define matrix A as follows:

A =

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Ideas:

In the long-run, on each day,

Oscar gets **3** emails.

Kyle gets **3** emails.

Isabel gets **2** emails.

Heather gets **3** emails.

$$\begin{matrix} O \\ K \\ I \\ H \end{matrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

O K I H

O = K' + H' + I' → Oscar receives all emails from Kyle, Heather, and Isabel on the previous day.

K = O' + H' + I'

I = I' + H'

H = K' + H + I'

O, K, I, H represent the number of emails that each person has today.

O', K', I', H' represent the number of emails that each person had yesterday.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} O' \\ K' \\ I' \\ H' \end{bmatrix} = \begin{bmatrix} O \\ K \\ I \\ H \end{bmatrix}$$

$$\lim_{n\to\infty} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is the starting condition on day 1

Definitions and Observations:

Each person i has a friend vector vi = sequence of 0s and 1s:

0 if i does not send email to j

1 if i does send email to j

$$\begin{bmatrix} O \\ K \\ I \\ H \end{bmatrix}_{\text{next day}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} O \\ K \\ I \\ H \end{bmatrix}_{\text{previous day}}$$

where the i-th column is person i's friendship column

x' = Ax, where x' = next day (easy computation)

What happens with $A^n x_0$ as n approaches infinity?

$A^n x_0$ represents the count/number of emails on day n.

We're actually after proportions, which is $\frac{A^n x_0}{T_n}$, where T_n is the total number of emails on day n.

Eigenvector & Eigenvalue Definition:

For a matrix A, when applied on a vector, the result is a scalar multiple

Define an eigenvector of A s.t. $Av = \lambda v$, $\lambda \in F$

A = operator

λ = eigenvalue

v = eigenvector

Wednesday, May 8 (Lecture 2)

Recall:

↳ Def: Vector space V, $T \in L(V)$, $v \in V$, $\lambda \in \mathbb{F}$

v is an eigenvector of T w/ corresponding eigenvalue λ if $Tv = \lambda v$ and $v \neq 0$ (bc $T \cdot 0 = \lambda \cdot 0$ is always true, but silly)

↳ (λ, v) is eigenpair of T

Q: Can one eigenvalue have multiple eigenvectors? (1) How about vice-versa? (2)

A: (1) Yes, an eigenvalue can have multiple eigenvectors

↳ Kye notes:

↳ Claim: for a given λ, the set $\{v \mid Tv = \lambda v\}$ is closed under scalar multiplcation

↳ Brandon Proof:

Let λ and v be eigenpair of T. Let $\alpha \in \mathbb{F}$ where $\alpha \neq 0$. Consider $v' = \alpha v$. Then $T(v') = T(\alpha v) = \alpha T(v) = \alpha(\lambda v) = \lambda(\alpha v) = \lambda v'$

↳ Kye Q:

↳ Is $\{v \mid Tv = \lambda v\}$ closed under addition?

↳ Break into groups:

↳ let $v \in V$ s.t. $T(v) = \lambda v$

let $u \in V$ s.t. $T(u) = \lambda u$

$$T(u+v) = \lambda(u+v) = \lambda u + \lambda v = T(u) + T(v) \quad \square$$

→ λ is fixed here, and can have many eigenvect

↳ Josie Proof:

↳ let (λ, u) and (λ, v) be eigenpairs of T, let $v' = v + u$

$$T(v') = T(u+v) = T(u) + T(v) = \lambda u + \lambda v = \lambda(u+v) = \lambda v' \quad \square$$

↳ Takeaway:

↳ $\{v \mid Tv = \lambda v\}$ is a subspace called the eigenspace of λ (or λ-eigenspace)

(2) No, each eigenvector corresponds to exactly one eigenvalue

↳ no proof done here

Q: How do we find eigenpairs?

Given T is the form of a matrix, compute $\det(T - \lambda I)$ to get a polynomial in λ. Find the roots/zeros of this polynomials.

Claim: The roots are the eigenvalues of T

ex: $T = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ Find eigenvalues of T

$$\det\left(\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2-\lambda & 2 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = (2-\lambda)(2-\lambda)(3-\lambda) \Rightarrow \lambda = 2, 3$$

"charistaristic polynomial of T" → $\chi_T(x) = \det(T - \lambda I) = (2-x)^2(3-x)$

Def: Given a polynomial p and root r, i.e. $p(r)=0$, the multiplicity of r is the number of times $x-r$ appears in the factorization of p (into monomials)

Q₁: Fix λ, are there independent λ-eigenvectors (can there be?)
↳ no proof done here

Q₂: Prove $\chi_T(\lambda)=0 \iff \lambda$ is eigenvalue of T

Nikhil Claim: Yes, and we can further prove that all of the λ-eigenvectors are the vectors in $\ker(T-\lambda I)$

Nikhil Proof: Using the definition of an eigenvector we can show that its in the kernel

$$Tv = \lambda v$$
$$(T-\lambda I)v = 0 \implies v \in \ker(T-\lambda I)$$

Since all λ-eigenvectors are non-zero, $\ker(T-\lambda I) \neq \{0\}$. Then, we know that $\det(T-\lambda I)=0$, which can have multiple solutions (proof cut a bit short)

Kye Final Q: What does $\ker(T-\lambda I)$ have to do with eigenvectors?

Thursday, May 9 (Discussion 2)

Recall:

- Definitions:
  - Given $T: L(V)$, $\lambda \in \mathbb{F}$, λ-eigenspace of T is $\{v \mid Tv = \lambda v\}$
  - Characteristic Polynomial of T: $\chi_T(z) = \det(zI - A)$
    - property: $\chi_T(z) = 0 \iff z$ is eigenvalue of T * Kye: partially proven but can you do on your own?
  - Given polynomial p, root r ($p(r)=0$) multiplicity of r is the # of times $(x-r)$ occurs in factorization of $p(x)$ into monomials
  - If λ is an eigenvalue of T, multiplicity of λ is multiplicity of λ as a root of χ_T * algebraic definition
  - The number of lin ind λ-eigenvectors is the multiplicity of λ * geometric definition

What is $\text{Ker}(\lambda I - T)$?

- conjecture: $\text{Ker}(\lambda I - T)$ is the λ-eigenspace

Claim: To prove two sets A and B are equal, prove $A \subseteq B$ and $B \subseteq A$, i.e.:

- show $\forall x \in A$, $x \in B$
- show $\forall y \in B$, $y \in A$

Proof:

- Claim: $\text{Ker}(\lambda I - T) = \lambda$-eigenspace
- Known: $U = \lambda$-eigenspace $= \{v \mid Tv = \lambda v\}$
  $S = \text{Ker}(\lambda I - T) = \{v \mid (\lambda I - T)v = 0\}$
- WTS: $S \subseteq U$ and $U \subseteq S$

Given some $T \in \mathbb{F}^{m \times n}$ and $\lambda \in \mathbb{F}$:

$S \subseteq U$	$U \subseteq S$
Let $v \in S$, then $(\lambda I - T)v = 0$	Let $u \in U$, then $Tu = \lambda u$
$(\lambda I - T)v = 0$	$Tu = \lambda u$
$\lambda I v - Tv = 0$	$Tu - \lambda u = 0$
$\lambda v - Tv = 0$	$Tu - \lambda(Iu) = 0$
$\lambda v = Tv$	$Tu - \lambda I u = 0$
$v \in U$ □	$u \in S$ □

ex: Find the eigenvalues and their corresponding λ-eigenspaces for the following matrices

a) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

a) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\lambda: 2, 2$ $\quad \text{Ker}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = \text{Ker}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \to \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

↑ $\det(T - \lambda I)$, shortcut since upper triangular so $\chi_T(\lambda) = (2-\lambda)(2-\lambda)$

b) $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ $\lambda = 1, 1$

$\uparrow$
$\chi_T(\lambda) = (1-\lambda)(1-\lambda)$

$\ker\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} - \begin{bmatrix}1&1\\0&1\end{bmatrix}\right) = \ker\left(\begin{bmatrix}0&-1\\0&0\end{bmatrix}\right)$

$\begin{bmatrix}0&-1\\0&0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}-b\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

Note: This question shows the difference between the algebraic and geometric definition of multiplicity. They are both correct, just different

c) $\begin{bmatrix}0&1\\1&1\end{bmatrix}$ $\det\left(\begin{bmatrix}-\lambda&1\\1&1-\lambda\end{bmatrix}\right) = -\lambda(1-\lambda)-1 = \lambda^2-\lambda-1 \rightarrow \lambda = \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}$

$\lambda = \frac{1-\sqrt{5}}{2}$ $\ker\left(\begin{bmatrix}0&1\\1&1\end{bmatrix} - \begin{bmatrix}\frac{1-\sqrt{5}}{2}&0\\0&\frac{1-\sqrt{5}}{2}\end{bmatrix}\right) = \ker\left(\begin{bmatrix}\frac{\sqrt{5}-1}{2}&1\\1&1+\frac{\sqrt{5}-1}{2}\end{bmatrix}\right) \rightarrow \begin{bmatrix}\frac{\sqrt{5}-1}{2}&1\\1&1+\frac{\sqrt{5}-1}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\frac{\sqrt{5}-1}{2}x + y = 0$
$x + y\left(1+\frac{\sqrt{5}-1}{2}\right) = 0$
$\downarrow$
$x = \frac{-1-\sqrt{5}}{2}$
$y = 1$

$\begin{bmatrix}\frac{-1-\sqrt{5}}{2}\\1\end{bmatrix}$

$\lambda = \frac{1+\sqrt{5}}{2}$ $\ker\left(\underbrace{\begin{bmatrix}\frac{-1-\sqrt{5}}{2}&1\\1&1+\frac{-1-\sqrt{5}}{2}\end{bmatrix}}_{T-\lambda I}\right) \rightarrow \begin{bmatrix}\frac{-1-\sqrt{5}}{2}&1\\1&1+\frac{-1-\sqrt{5}}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$\left.\begin{aligned}-\tfrac{1-\sqrt{5}}{2}x + y &= 0\\ x + y\left(1+\tfrac{-1-\sqrt{5}}{2}\right) &= 0\end{aligned}\right\}$ $x = \frac{\sqrt{5}-1}{2}$, $y = 1$ $\begin{bmatrix}\frac{\sqrt{5}-1}{2}\\1\end{bmatrix}$

$\operatorname{span}\left\{\begin{bmatrix}\frac{-1-\sqrt{5}}{2}\\1\end{bmatrix}, \begin{bmatrix}\frac{-1+\sqrt{5}}{2}\\1\end{bmatrix}\right\}$

Def: multiplicity of eigenvalue λ: $\dim(\lambda\text{-eigenspace}) = \dim(\ker(\lambda I - T))$ *geometric def

Question from Lecture 2: Can one eigenvalue have multiple linearly independent eigenvectors?
Answer: Yes

Kye Final Q: What are possible dimensions of eigenspaces?

Friday, May 9 (Lecture 3)

Inverses: Let V be an n-dim $\mathbb{F}$-vec space and $T \in L(V) := L(V,V)$. Suppose $A = {}_B[T]_B$ for some basis B of V.

T is bijective is equivalent to:
- there exists a $T' \in L(V)$ s.t. $TT' = T'T = I_V$ $\quad (I_V(v) := v)$
  - inverse map present on HW4 A1c
- there exists an $A' \in \mathbb{F}^{n \times n}$ s.t. $AA' = A'A = I_{\mathbb{F}^{n\times n}}$ $\quad \left(I_{\mathbb{F}^{n\times n}} := \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right)$
  - inverse matrix present on HW 5 A4c
  - $A = {}_B[T]_B$
  - $A' = {}_B[T']_B$

Fact: In either case, the inverse is unique $\longrightarrow$ ${}_B[T^{-1}]_B = {}_B[T]_B^{-1}$

Proposition: T^{-1} exists IFF $\det(A) \neq 0$

Proof: T^{-1} exists $\Rightarrow A^{-1}$ exists
$\Rightarrow \det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$
*HW5 A4c
$\Rightarrow \det(A) \neq 0$

T^{-1} does not exist $\Rightarrow \operatorname{rank}(T) < n$ * no inverse means not bijective (either not injective or surjective), so $\operatorname{rank}(T) < n$
$\Rightarrow$ col. rank of $A < n$
*HW 4 A3d
$\Rightarrow$ columns are linearly dependent
$\Rightarrow \det(A) = 0$
*quiz 5 problem 1

Eigenvalues and Eigenvectors: Let V be an $\mathbb{F}$-vec space and $T \in L(V)$

Def: If $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$ and $v \in V \setminus \{0\}$, then λ is an eigenvalue and v is an eigenvector of T

Equivalently: $v \in \ker(\lambda I - T) \setminus \{0\}$. Thus, λ is an eigenvalue of T IFF $\lambda I - T$ is NOT injective, in which case the non-zero vectors in the eigenspace $E_T(\lambda) := \ker(\lambda I - T)$ are the eigenvectors of T with eigenvalue λ

Finite-Dimensions: If V is n-dim, we can test the injectivity of $\lambda I - T$ using the determinant because injectivity in this case is equivalent to bijectivity

Def: The characteristic polynomial of T is $\chi_T(\lambda) := \det(\lambda I - [T])$
↑ matrix of T

Rmk: Any basis can be used for T (why?)
- χ_T is a polynomial of degree n * quiz 5 problem 2