DETERMINANTS

Let $A \in \mathbb{F}^{n\times n}$. We will write
$a_j \in \mathbb{F}^n$ for the j^{th} col. of A,
$a_{ij} \in \mathbb{F}$ for the (i,j)-entry of A.

- <u>Def</u> The DETERMINANT is the function $\det : \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}$ s.t.

(i) $\det(a_1, \dots, a_{j-1}, \alpha a_j + \beta b_j, a_{j+1}, \dots, a_n)$
$= \alpha \det(\quad " \quad , \quad a_j \quad , \quad " \quad)$
$+ \beta \det(\quad " \quad , \quad b_j \quad , \quad " \quad)$
[MULTILINEAR]

(ii) $\det(a_1, \dots, a_n) = 0$ if $a_i = a_j$ for some $i \neq j$
[ALTERNATING]

(iii) $\det(e_1, \dots, e_n) = 1$, where
$e_j := (0, \dots, 0, 1, 0, \dots, 0)$
[NORMALIZED]

Rmk For now, assume that such a fn. exists and is unique.

Prop $\det(\ldots, a_i, \ldots, a_j, \ldots)$
$= -\det(\ldots, a_j, \ldots, a_i, \ldots)$

Pf
$$\begin{aligned}
0 &= \det(\ldots, a_i+a_j, \ldots, a_i+a_j, \ldots) \\
&= \det(\ldots, a_i, \ldots, a_i+a_j, \ldots) \\
&\quad + \det(\ldots, a_j, \ldots, a_i+a_j, \ldots) \\
&= \det(\ldots, a_i, \ldots, a_j, \ldots) \\
&\quad + \det(\ldots, a_j, \ldots, a_i, \ldots)
\end{aligned}$$

When $n=2$, we have

$$\det(a_1, a_2) = \det(a_{11}e_1 + a_{21}e_2, a_{12}e_1 + a_{22}e_2)$$

$$a_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad a_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$\begin{aligned}
&= \det(a_{11}e_1, a_{12}e_1) + \det(a_{11}e_1, a_{22}e_2) + \\
&\quad \det(a_{21}e_2, a_{12}e_1) + \det(a_{21}e_2, a_{22}e_2) \\
&= a_{11}a_{22}\det(e_1, e_2) + a_{21}a_{12}\det(e_2, e_1)
\end{aligned}$$

$= a_{11}a_{22} - a_{21}a_{12}$

$n=3$

$$+\underset{1}{a_{11}}\,\underset{2}{a_{22}}\,\underset{3}{a_{33}} - \underset{2}{a_{21}}\,\underset{1}{a_{12}}\,\underset{3}{a_{33}} + \underset{3}{a_{31}}\,\underset{1}{a_{12}}\,\underset{2}{a_{23}}$$
$$-\underset{1}{a_{11}}\,\underset{3}{a_{32}}\,\underset{2}{a_{23}} + \underset{2}{a_{21}}\,\underset{3}{a_{32}}\,\underset{1}{a_{13}} - \underset{3}{a_{31}}\,\underset{2}{a_{22}}\,\underset{1}{a_{13}}$$

In general,

$$\det(A) = \sum_\sigma \pm\, a_{\sigma(1),1} \cdot a_{\sigma(2),2} \cdots a_{\sigma(n),n}$$

where σ is a PERMUTATION of $\{1, \dots, n\}$ and

$\pm$ depends on the # of swaps to change $\sigma(1), \dots, \sigma(n)$ to $1, \dots, n$ (+ for an even #, − for an odd #)

do diff. seq. of swaps have the same parity?...

<u>Rmk</u> We can check that this fn. ("Leibniz formula") satisfies (i)–(iii); the argument above shows that it is unique.

Let $A := [a_1 \cdots a_n] \in \mathbb{F}^{n \times n}$,
$a_j := (a_{1j}, \ldots, a_{nj}) \in \mathbb{F}^n$.

Def The DETERMINANT is the function $\det : \mathbb{F}^{n \times n} \to \mathbb{F}$ that is (i) MULTILINEAR, (ii) ALTERNATING, and (iii) NORMALIZED.

Prop If A is of the form

"upper triang."
$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{bmatrix},$$
then

$$\det(A) = a_{11} \cdots a_{nn}.$$

Pf $\det(A) = \sum_\sigma \pm a_{\sigma(1),1} \cdots a_{\sigma(n),n}$

If $\sigma(j) \leq j$ for all j, then

$\sigma(1) = 1 \Rightarrow \sigma(2) = 2 \Rightarrow \cdots \Rightarrow \sigma(n) = n.$

Otherwise, $\sigma(j) > j$ for some j, so $a_{\sigma(j),j} = 0$.

Hence $\det(A) = a_{1,1} \cdots a_{n,n}$.

Def A POLYNOMIAL over $\mathbb{F}$ is an expression of the form $p(x) := \sum_{i=0}^{n} \alpha_i x^i$, $\alpha_i \in \mathbb{F}$.
Its DEGREE $\deg(p)$ is the largest i s.t. $\alpha_i \neq 0$.

Rmk If all the α_i are 0, we sometimes use the convention that $\deg(p) = -\infty$.

Rmk
$$\deg(p+q) \leq \max\{\deg(p), \deg(q)\}$$
$$\deg(\alpha p) = \deg(p), \quad \alpha \neq 0$$
$$\deg(pq) = \deg(p) + \deg(q)$$

Thm (Euclidean division). If $a, b \in$ $P(\mathbb{F})$ and $b \neq 0$, there exist unique $q, r \in P(\mathbb{F})$ s.t. $a = bq + r$ and $\deg(r) < \deg(b)$.

poly. over $\mathbb{F}$

When is $p(x)$ divisible by $x - c$ $(c \in \mathbb{F})$? That is, when is there 0 remainder?

Ex $x^2 - 1 = (x-1)(x+1)$
so $x-1, x+1 \mid x^2 - 1$.

Ex $x^2 - 1 = (x-2)(x+2) + 3$
so $x-2, x+2 \nmid x^2 - 1$.

Since $p(x) = (x-c)\,q(x) + r, \quad r \in \mathbb{F}$
$\Rightarrow p(c) = r$

we have $r = 0 \Leftrightarrow p(c) = 0$.