MATH 115A Week 4 Notes

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Lecture 1: Monday, April 21

Exercise – Problem B3 from Homework 2

(a) The following two groups:

$$(\mathbb{Z}_4, +_4) \quad \text{and} \quad (\mathbb{Z}_5 \setminus \{0\}, \cdot_5).$$

are secretly "the same". How so?

(b) Provide a precise definition of what it means for two groups to be "the same": given groups $(G, \heartsuit)$ and $(G', \diamondsuit)$ are the same if there is a function $f : G \to G'$ such that ... ?

Question: What does it mean for two groups to be "the same"?
Idea:

$$(G, \heartsuit) \cong (G', \diamondsuit)$$

if there exists a bijective mapping $f : G \to G'$ (a relabeling of elements) such that

$$\text{for all } g_1, g_2 \in G, \quad f(g_1 \heartsuit g_2) = f(g_1) \diamondsuit f(g_2).$$

Here, bijective means both injective and surjective.

Definition (Isomorphism): Two vector spaces V and V' are called "the same" if there is a bijective linear map

$$f : V \to V',$$

satisfying

$$f(v_1 + v_2) = f(v_1) + f(v_2), \quad f(cv) = c\, f(v)$$

for all $v_1, v_2 \in V$ and scalars c. Such an f is called an *isomorphism.* If one exists, V and V' are *isomorphic*, denoted $V \cong V'$.

Examples: Determine which (if any) of these spaces are isomorphic:

- $\mathbb{R}^{2\times 2}$,
- $\{(w, x, y, z) : w + x + y + z = 0\}$,
- $\mathcal{L}(\mathbb{R}^3, \mathbb{R})$ (the space of linear maps $\mathbb{R}^3 \to \mathbb{R}$),
- $P_3(\mathbb{R})$,
- $\mathbb{C}$ with Real Scalars.

Conjectures:

$$\begin{aligned}
\dim \mathcal{L}(\mathbb{R}^3, \mathbb{R}) &= 3,\\
\dim\{(w,x,y,z) : w+x+y+z=0\} &= 3,\\
\dim\left(\mathbb{R}^{2\times 2}\right) &= 4,\\
\dim P_3(\mathbb{R}) &= 4,\\
\dim \mathbb{C} &= 2.
\end{aligned}$$

If two vector spaces over the same field have equal dimension, then they are isomorphic. Note: spaces (1) and (2) are not isomorphic despite having the same dimension.

Discussion 1: Tuesday, April 22

Question 1: If $\dim(V_1) = \dim(V_2)$ for two vector spaces, must they be isomorphic?
Answer: Only if they are defined over the same field.

Recall: An isomorphism of vector spaces is a bijective linear map $T : V \to W$, and both V and W must share the same scalar field (e.g., $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}_2$).

Proof: Assume V, W are vector spaces. Let $B_V = \{v_1, \ldots, v_n\}$ and $B_W = \{w_1, \ldots, w_n\}$ be bases for V and W, respectively. Define T by $T(v_i) = w_i$ and extend linearly. Then for any

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i, v \in V$$

we have

$$T(v) = \sum_{i=1}^{n} \alpha_i T(v_i) = \sum_{i=1}^{n} \alpha_i w_i.$$

Surjectivity follows since B_W spans W. By rank–nullity,

$$\dim V = \dim W = n,\ \dim V = \dim(\operatorname{im} T) + \dim(\ker T) = \dim W + \dim(\ker T),$$

so $\dim(\ker T) = 0$, then $\ker T = \{0\}$ and T is injective. Hence T is an isomorphism.

Theorem. Every n-dimensional vector space V over a field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$. Given a basis $\{b_1, \ldots, b_n\}$, we write

$$(x_1, \ldots, x_n) \in V \leftrightarrow x_1 b_1 + \cdots + x_n b_n.$$

Additional note: Two linear maps $f, g : V \to W$ are equal if $f(x) = g(x)$ for all $x \in V$.

Question 2: Is $\dim L(\mathbb{R}^3, \mathbb{R}) = 3$?
A basis for this space is given by the coordinate functionals

$$f_1(x,y,z) = x, \quad f_2(x,y,z) = y, \quad f_3(x,y,z) = z,$$

so any linear map can be written as

$$f(x,y,z) = \alpha_1 x + \alpha_2 y + \alpha_3 z.$$

Proof. Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}$ is linear. Let the basis $B_L = \{f_1, f_2, f_3\}$ where

$$f_1(x,y,z) = x, \quad f_2(x,y,z) = y, \quad f_3(x,y,z) = z.$$

Then

$$\begin{aligned} f(x,y,z) &= f\big((x,0,0) + (0,y,0) + (0,0,z)\big) \\ &= f\big(x(1,0,0) + y(0,1,0) + z(0,0,1)\big) \\ &= x\, f(1,0,0) + y\, f(0,1,0) + z\, f(0,0,1) \\ &= \alpha_1\, f_1(x,y,z) + \alpha_2\, f_2(x,y,z) + \alpha_3\, f_3(x,y,z) \\ &= f(x,y,z). \end{aligned}$$

Isomorphism is possible when V and W have the same field and $\dim(V) = \dim(W)$. We denote this by $V \cong W$.

To prove $V \cong W$, construct a linear map $T\colon V \to W$ and show that T is both injective and surjective.

Corollary. If V is n-dimensional with basis $\{b_1, b_2, \ldots, b_n\}$, then writing

$$[v] = (x_1, x_2, \ldots, x_n)$$

as the coordinate vector of $v \in V$ means

$$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n.$$

$\square$

Lecture 2: Wednesday, April 23

Matrices

Warm-up Suppose $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$ satisfies:

$$T(0,0,1) = 2, \quad T(0,1,1) = 3, \quad T(1,1,1) = 3.$$

Find $T(3,2,1)$.

Since T is linear, T is additive and homogeneous

$$T(3,2,1) = T\big(3(1,1,1) - (0,1,1) - (0,0,1)\big) = 3T(1,1,1) - T(0,1,1) - T(0,0,1) = 3 \cdot 3 - 3 - 2 = 4.$$

More generally, for $(c_0, c_1, c_2) \in \mathbb{R}^3$,

$$\begin{aligned} T(c_0,0,0) &= T\big(c_0(1,0,0)\big) = c_0\, T\big((1,1,1) - (0,1,1)\big) = c_0\big(T(1,1,1) - T(0,1,1)\big) = c_0(3-3) = 0, \\ T(0,c_1,0) &= T\big(c_1(0,1,0)\big) = c_1\, T\big((0,1,1) - (0,0,1)\big) = c_1\big(T(0,1,1) - T(0,0,1)\big) = c_1(3-2) = c_1, \\ T(0,0,c_2) &= T\big(c_2(0,0,1)\big) = c_2\, T(0,0,1) = 2c_2, \\ T(c_0,c_1,c_2) &= T(c_0,0,0) + T(0,c_1,0) + T(0,0,c_2) = c_1 + 2c_2. \end{aligned}$$

Matrix Representation of a Linear Map

Let V be an n-dimensional vector space over a field $\mathbb{F}$ with basis $B_V = \{v_1, \ldots, v_n\}$, and W an m-dimensional space with basis $B_W = \{w_1, \ldots, w_m\}$. If $T : V \to W$ is linear, then for any $v \in V$ we have

$$v = \sum_{j=1}^{n} \beta_j v_j$$

for some $\beta_j \in \mathbb{F}$. This implies that

$$Tv = \sum_{j=1}^{n} \beta_j T v_j$$

by the linearity of T (where $Tv, Tv_j \in W$).

For each j, we have

$$Tv_j = \sum_{i=1}^{m} \alpha_{ij} w_i$$

for some $\alpha_{ij} \in \mathbb{F}$, which implies that

$$Tv = \sum_j \beta_j (\sum_i \alpha_{ij} w_i) = \sum_i (\sum_j \alpha_{ij} \beta_j) w_i$$

Definition. The *matrix of* $T \in \mathcal{L}(V, W)$ *with respect to* (B_V, B_W) is the $m \times n$d matrixd

$$[T]_{B_V}^{B_W} = \left(\alpha_{ij}\right)_{1 \le i \le m,\, 1 \le j \le n},$$

whose jth column lists the coordinates of $T(v_j)$ in B_W.

Examples

- **Rotation in $\mathbb{R}^2$.** With the standard basis $e_1 = (1, 0)$, $e_2 = (0, 1)$, the linear map rotating counterclockwise by angle θ satisfies

  $$T(e_1) = (\cos\theta, \sin\theta), \quad T(e_2) = (-\sin\theta, \cos\theta),$$

  so

  $$[T] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- **Differentiation on $P_3(\mathbb{R})$.** Let $D(p) = p'$ and choose the basis $\{1, x, x^2, x^3\}$. Then

  $$D(1) = 0, \quad D(x) = 1, \quad D(x^2) = 2x, \quad D(x^3) = 3x^2,$$

  giving the matrix

  $$[D] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

  Now suppose $T_1 \in L(V, W)$ has matrix $[\alpha_{ij}]$ and $T_2 \in L(V, W)$ has matrix $[\beta_{ij}]$ with basis B_v, B_u.

  Note $T_1 + T_2 \in L(U, W)$ since $T_1 v_j = \sum_i \alpha_{ij} w_i$ and $T_2 v_j = \sum_i \beta_{ij} w_i$ we have $(T_1 + T_2) v_j = \sum_i (\alpha_{ij} \beta_{ij}) w_i$

  Now suppose $S \in L(U, V)$ with matrix $[\beta_{ij}]$ $T/inL(V, W)$ with matrix $[\alpha_{ij}]$, U is a p-dimensional vector space with basis $B_u u_1, u_2, ..., u_n$ Note: $T \circ S \in L(U, W)$

  Ending QuestioN: What is the matrix with respect to B_v, B_u?

Discussion 2: Thursday, April 24

Coordinate and Matrix Representations

Let V have basis $B_V = \{v_1, \ldots, v_n\}$. Any vector $v \in V$ has coordinates

$$[v]_{B_V} = (a_1, \ldots, a_n), \quad v = a_1 v_1 + \cdots + a_n v_n.$$

If W has basis $B_W = \{w_1, \dots, w_m\}$ and $T : V \to W$ is linear, then

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i, \quad j = 1, \dots, n,$$

so the matrix of T is the $m \times n$ array

$$[T]_{B_V}^{B_W} = (\alpha_{ij})_{1 \le i \le m,\, 1 \le j \le n}.$$

Moreover, for any $v \in V$,

$$[T]_{B_V}^{B_W} \, [v]_{B_V} = [T(v)]_{B_W}.$$

Example: Differentiation

Define $D : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ by $D(p) = p'$. For instance,

$$D[x^2 - 2x] = 2x - 2.$$

Choose bases

$$B_{P_3} = \{1, x, x^2, x^3\}, \quad B_{P_2} = \{1, x, x^2\}.$$

Then

$$D(1) = 0, \qquad D(x) = 1, \qquad D(x^2) = 2x, \qquad D(x^3) = 3x^2,$$

and the matrix of D is

$$[D] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Example: Evaluation Map

Define $E : P_2(\mathbb{R}) \to \mathbb{R}^2$ by

$$E(p) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}.$$

With bases $B_{P_2} = \{1, x, x^2\}$ and the standard basis $B_{\mathbb{R}^2} = \{(1,0), (0,1)\}$, we compute

$$E(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad E(x) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad E(x^2) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Hence the matrix of E is

$$[E] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

Composition $E \circ D$

The composition $E \circ D : P_3(\mathbb{R}) \to \mathbb{R}^2$ has matrix

$$[E \circ D] = [E]\,[D] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 12 \end{pmatrix}.$$

This follows from the associative property of composition and matrix multiplication. A proof from first principles uses

$$[E \circ D](p) = E(D(p)),$$

and the coordinate relations for each basis element.

Lecture 3: Friday, April 25

Let $T \in L(V, W)$, where V is an n-dimensional vector space with basis $B_V = \{v_1, \ldots, v_n\}$ and W is m-dimensional with basis $B_W = \{w_1, \ldots, w_m\}$. For each j, write

$$T(v_j) = \sum_{i=1}^{m} \beta_{ij}\, w_i.$$

Then the *matrix of* T *with respect to* (B_V, B_W) is

$${}_{B_W}[T]_{B_V} = (\beta_{ij})_{1\le i\le m,\, 1\le j\le n} \in F^{m\times n}.$$

Example: Rotation in $\mathbb{R}^2$

Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation by angle θ. With standard basis $\{(1,0),(0,1)\}$,

$$[T_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Composition of Linear Maps

If $S \in L(U, V)$ with basis $B_U = \{u_1, \ldots, u_p\}$ and $T \in L(V, W)$ as above, then

$${}_{B_W}[T \circ S]_{B_U} =_{B_W} [T]_{B_V}\ {}_{B_W}[S]_{B_U}.$$

In particular, for $T = T_\theta$,

$$[T_\theta \circ T_\theta] = [T_\theta]^2,$$

and since $T_\theta \circ T_\theta = T_{2\theta}$, one recovers the double-angle formulas.

Example: Kye

Let

$$B = \{1, x, x^2\}, B' = \left\{ \tfrac{x(x-1)}{2},\ -(x+1)(x-1),\ \tfrac{x(x+1)}{2} \right\}.$$

If a polynomial p has coordinates $[p]_B = (\beta_0, \beta_1, \beta_2)^T$ and $[p]_{B'} = (\beta'_0, \beta'_1, \beta'_2)^T$, then what are the β's in terms of the β''s (and vice versa)?

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \beta'_0 \\ \beta'_1 \\ \beta'_2 \end{bmatrix} = \beta'_0 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \beta'_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \beta'_2 \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Consider $I \in \mathcal{L}(P_2(\mathbb{R}), P_2(\mathbb{R}))$ given by

$$(Ip)(x) := p(x) \text{ [Identity Map]}$$

then,

$${}_B[I]_{B'} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$