Math 115A Week 3 Scribe Notes

LECTURE 1 | *Monday, 4/14*

Notation (for $\mathbb{F}$ being a field)

- $\mathbb{F}^{mxn}$: mxn matrices with entries in $\mathbb{F}$
- $\mathbb{F}^{\mathbb{N}}$: sequences with terms in $\mathbb{F}$
  - Ex: $(x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{F}$
- C(X): continues functions $f: X \to \mathbb{R}$
  - X is some set
  - Satisfy pointwise addition and scalar multiplication
    - $(f+\alpha g)(x) = f(x)+\alpha g(x)$
- $C^{\infty}(X)$: continuous AND infinitely differentiable (smooth) functions $f: X \to \mathbb{R}$

Definitions: Let $f: X \to Y$ be a function (for set X and set Y). f is:

- **Injective** if $f(x_1)=f(x_2)$ implies $x_1=x_2$
  - Every reachable output has a unique input (only one)
- **Surjective** if $\forall y \in Y, \exists x \in X: f(x) = y$
  - Every output is reachable (has an input)
- **Bijective** if f is both injective and surjective
  - Every output is reachable by a unique input

Exercise 1 (Testing Linear Combinations and Injectivity/Surjectivity):

Everyone in class is given a function on a sheet of paper and must answer:

1) What happens when you apply the function to a sum, scalar multiple, and linear combination of vectors?
2) Is the function injective, surjective, and/or bijective?

Solutions to Exercise 1 (for a sample function):

Sample function:

- $T : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$
- $(T(f))(x) := \frac{d}{dx}(f(x)) - f(x)$

Questions:

1) (a) Check sum

$(T(f+g))(x) = \frac{d}{dx}(f(x) + g(x)) - (f(x) + g(x))$

$(T(f+g))(x) = (\frac{d}{dx}f(x) - f(x)) + (\frac{d}{dx}g(x) - g(x)$

<u>$(T(f+g))(x) = (T(f))(x) + (T(g))(x)$</u>

(b) Check scalar multiplication

$(T(\propto f))(x) = \frac{d}{dx}(\propto f(x)) - \propto f(x)$

$(T(\propto f))(x) = \propto(\frac{d}{dx}f(x) - f(x))$

<u>$(T(\propto f))(x) = \propto(T(f))(x)$</u>

(c) Check linear combinations

$(T(\propto_1 f + \propto_2 g))(x) = \frac{d}{dx}(\propto_1 f(x) + \propto_2 g(x)) - (\propto_1 f(x) + \propto_2 g(x))$

$(T(\propto_1 f + \propto_2 g))(x) = \propto_1(\frac{d}{dx}f(x) - f(x)) + \propto_2(\frac{d}{dx}g(x) - g(x))$

<u>$(T(\propto_1 f + \propto_2 g))(x) = \propto_1(T(f))(x) + \propto_2(T(g))(x)$</u>

2) (a) Check if T is injective

For $f = e^x$ and $g = 0$,

$(T(f))(x) = e^x - e^x = 0$

$(T(g))(x) = 0 - 0 = 0$

Thus, because inputting f and g produce the same output, <u>T is not injective</u>

(b) Check if T is surjective

(Didn't have time to work on or discuss this proof)

(c) Check if T is bijective

Since T is not injective, <u>T is not bijective</u>

<u>Definition (Linear Map):</u>

A **map** $T:V\rightarrow W$ (for V and W being vector spaces over $\mathbb{F}$) is **LINEAR** if:

1) $T(v_1+v_2+...+v_n) = T(v_1) + T(v_2) + \ldots + T(v_n)$ for $v_1, ..., v_n \in V$
2) $T(\propto v) = \propto T(v)$, for $\propto \in \mathbb{F}$ and $v \in V$

Which is equivalent to saying

3) $T(\propto_1 v_1 + \propto_2 v_2 + \ldots + \propto_n v_n) = \propto_1 T(v_1) + \propto_2 T(v_2) + \ldots + \propto_n T(v_n)$ for $\propto_1, \ldots, \propto_n \in \mathbb{F}$ and $v_1, \ldots, v_n \in V$

<u>Proof (for equivalent statements):</u>

Suppose $T(\propto_1 v_1 + \ldots + \propto_n v_n) = \propto_1 T(v_1) + \ldots + \propto_n T(v_n)$

If every $\propto_i = 0$ except for $\propto_j$, then we have $T(\propto_j v_j) = \propto_j T(v_j)$, which is statement 1

If $\propto_1, \ldots, \propto_n = 1$, then we have $T(v_1 + \ldots + v_n) = T(v_1) + \ldots + T(v_n)$, which is statement 2

Thus, statement 3 is equivalent to saying statement 1 and statement 2 together

DISCUSSION 1 | *Tuesday, 4/15*

Recap of Lecture 1:

Definition (Linear Map)
- Given two vector spaces V, W
- f: V → W is linear if
  1. $\forall v_1, v_2 \in V;\ f(v_1 + v_2) = f(v_1) + f(v_2)$
  2. $\forall \lambda \in \mathbb{F}, \forall v \in V;\ f(\lambda v) = \lambda f(v)$

Exercise Set 1 (Testing Linearity of Linear Maps):

Approach: To test for linearity, check that the linear maps follow the (a) additive and (b) scalar multiplication operations defined above.

1. $f: \mathbb{R} \to \mathbb{R},\ f(x) = x + 1$
2. $f: \mathcal{P}_n(\mathbb{R}) \to \mathbb{R},\ f(p) = p(1)$
3. $f: C(\mathbb{R}) \to C(\mathbb{R}),\ [f(a)](x) = xa(x)$
4. $f: C(\mathbb{R}) \to C(\mathbb{R}),\ (f(a))(x) = x + a(x)$

SOLUTIONS to Exercise Set 1:
1. (a) Check Addition
   $x_1, x_2 \in \mathbb{R}$
   $f(x_1) = x_1 + 1$
   $f(x_2) = x_2 + 1$
   $f(x_1 + x_2) = (x_1 + x_2) + 1$
   $f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$
   Thus, since $f(x_1 + x_2) \neq f(x_1) + f(x_2)$, the function is **NOT** linear.
2. Isabel's Solution:
   (a) Check Addition
   Let $p_1 = a_1x^n + a_2x^{n-1} + ... + a_nx^0$ and $p_2 = b_1x^n + b_2x^{n-1} + ... + b_nx^0$
   $f(p_1 + p_2) = [(p_1 + p_2)(1)] = [((a_1 + b_1)x^n + (a_2 + b_2)x^{n-1} + .. (a_n + b_n)x^0)(1)]$
   $= (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n) = p_1(1) + p_2(1)$
   $= (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n)$
   (b) Check Scalar Multiplication
   $f(\lambda p_1) = \lambda f(p_1) = \lambda[(a_1x^n + a_nx^0)(1) = \lambda(a_1 + ... + a_n) = \lambda p_1(1)$
   Thus, since addition and scalar multiplication both hold, the function is **LINEAR**.
3. Robert's Solution:
   (a) Check Addition:
   $f(a) = xa(x)$

$f(a + b) = x(a + b)$
$f(a) + f(b) = xa + xb = x(a + b)$
(b) Check Multiplication:
$f(\alpha a) = x\alpha a(x)$
$\alpha f(a) = \alpha x a(x)$
Since $f(\alpha a) = \alpha f(a)$ and addition holds, this is **LINEAR**.

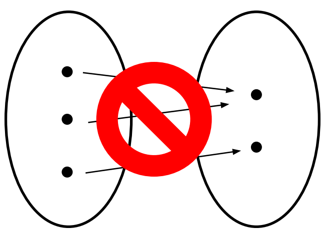
4. (a) Check Addition
$a, b \in C(\mathbb{R})$
$f(a + b)(x) = f(a + b)(x) = x + (a + b)(x) = x + ax + bx$
$f(a)(x) + f(b)(x) = (x + a(x)) + (x + b(x)) = 2x + a(x) + b(x)$
Since $f(a + b)(x) \not\simeq f(a)(x) + f(b)(x)$, the function is **NOT** linear.

Observation:

If f is linear, then $f(0) = 0$… maybe?

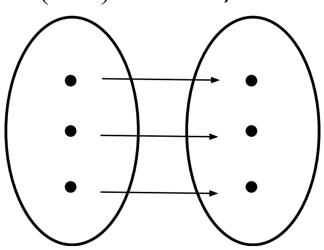
Review:

- **Injective** (one-to-one): If $f(x_1) = f(x_2)$, then $x_1 = x_2$



**Not injective since multiple inputs have the same output*

- **Surjective** (onto): $\forall b \in B,\ \exists\, a \in A$ s.t. $f(a) = b$



**Surjective since each output is "accounted for" or "mapped to"*

- **Bijective**: Both injective and surjective

Definition (Kernel)
- If $f: V \to W$ is linear, $ker f = \{a \in V:\ f(a) = 0\}$

Definition (Image)
- If $f: V \to W$ is linear, $im f = \{w \in W:\ \exists\, v \in V,\ f(v) = w\}$

Exercise Set 2:

Q: What is the kernel and image of each function? What are the dimensions of the kernel/image? What does kernel/image have to do with being injective/surjective?

1. $T\colon \mathcal{P}_3(\mathbb{R}) \to \mathbb{R};\ T(p) = p(1)$
2. $T\colon \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R});\ T(p) = \frac{\partial^2}{\partial x^2} p(x)$
3. $T\colon \mathbb{F}^n \to \mathbb{F}^{n-1};\ T(x_1, ..., x_n) = T(x_2, ..., x_n)$*

*Kye's suggestion is to start with the 3rd exercise first, as it is easier.

SOLUTIONS to Exercise Set 2:

1. Kernel: $kerT = \{(-a_1 - a_2 - a_3) + a_1 x + a_2 x^2 + a_3 x^3\}$
   Image: $imT = \mathbb{R}$
   Dimension: $dim(kerT) = 3, dim(imf) = 1$
2. Kernel: $kerT = \{\alpha_1 + \alpha_2 x,\ \alpha_1,\ \alpha_2 \in \mathbb{R}\}$
   Image: $imT = \mathcal{P}_{n-2}(\mathbb{R})$
   Dimension: $dim(kerT) = 2, dim(imT) = n - 2$
3. Kernel: $kerT = (k, 0, ..., 0),\ k \in \mathbb{R}$
   Image: $imT = \mathbb{F}^{n-1}$
   Dimension: $dim(kerT) = 1, dim(imT) = n - 1$

Ending Note

We must define the domain and codomain as *vector spaces* before finding the kernel/image!!

Ponder

What is the correlation between the dimension of the kernel and image? Ponder this until the next class…

LECTURE 2 | *Wednesday, 4/16*

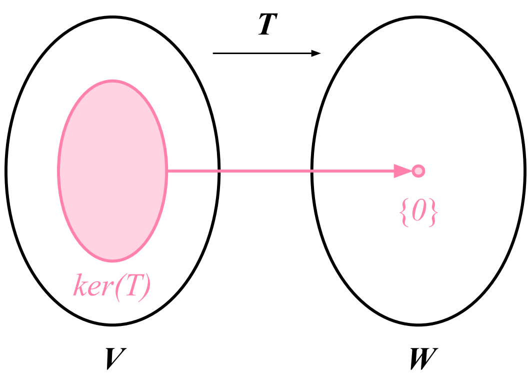
KERNEL AND IMAGE

Let V, W be vector spaces over $\mathbb{F}$ and $T \in L(V, W)$

- ★ $L(V, W)$ is the set of Linear maps from $V \to W$

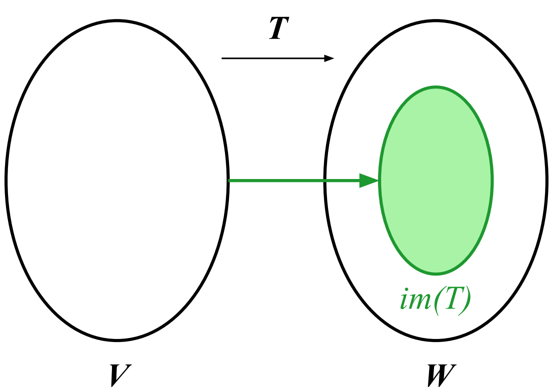
Define: The **"KERNEL"** of T is: $ker(T) \coloneqq \{v \in V : Tv = 0\} \subseteq V$

- ★ This can be read as "the kernel of T is all the inputs that map to 0"



Define: The **"IMAGE"** of T is: $im(T) \coloneqq \{Tv : v \in V\} \subseteq W$

- ★ This can be read as "the image of T is all the mappable outputs"



Example 1:

$T: P_4(\mathbb{R}) \to P_3(\mathbb{R})$

$(Tf)(x) \coloneqq f''(x)$

"When does $Tf = 0$?" $\Leftrightarrow$ "When does $f''(x) = 0$?"

$Tf = 0$
$\Rightarrow f''(x) = 0$
$\Rightarrow f'(x) = \alpha_1, \ \alpha_1 \in \mathbb{R}$
$\Rightarrow f'(x) = \alpha_1 x + \alpha_2, \ \alpha_1, \alpha_2 \in \mathbb{R}$
$ker(T) = \{\alpha_1 x + \alpha_2 : \alpha_1, \alpha_2 \in \mathbb{R}\} = P_1(\mathbb{R})$
↳ This is a vector space of P^4!

$dim(ker(T)) = dim(P_1(\mathbb{R})) = 2$
↳ More generally, $dim(P_n(\mathbb{R})) = n + 1$

If $f \in P_4(\mathbb{R})$, then $f(x) = \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$,
So $(Tf)(x) = (4\alpha_4 x^3 + 3\alpha_3 x^2 + 2\alpha_2 x + \alpha_1)' = 12\alpha_4 x^2 + 6\alpha_3 x + 2\alpha_2 \in P_2(\mathbb{R})$
↳ This shows that something in the image must be contained within $P_2(\mathbb{R})$

$$im(T) \subseteq P_2(\mathbb{R})$$

Conversely, if we have some $g(x) = \beta_2 x^2 + \beta_1 x + \beta_0 \in P_2(\mathbb{R})$,
Then taking $12\alpha_4 = \beta_2, \ 6\alpha_3 = \beta_1, 2\alpha_2 = \beta_0$ gives $Tf = g$
↳ If I have something in P_2, I can apply T and get anything in P_2

$$P_2(\mathbb{R}) \subseteq im(T)$$

↳ Hence, $im(T) = P_2(\mathbb{R})$, and $dim(im(T)) = 3$

Observations from Ex. 1:

- ★ $ker(T) = P_1(\mathbb{R})$ is a subspace of $V = P_4(\mathbb{R})$
- ★ $im(T) = P_2(\mathbb{R})$ is a subspace of $W = P_2(\mathbb{R})$
- ★ $dim(V) = dim(ker(T)) + dim(im(T))$ since $5 = 2 + 3$

Example 2:
$T: \mathbb{R}^{2x2} \to \mathbb{R}^{2x2}$

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) := \begin{bmatrix} a_{11} & \frac{1}{2}(a_{21}+a_{12}) \\ \frac{1}{2}(a_{21}+a_{12}) & a_{22} \end{bmatrix}$$

★ What are $ker(T),\ im(T)$?

★ Are they subspaces? If so, what dim?

Set $\begin{bmatrix} a_{11} & \frac{1}{2}(a_{21}+a_{12}) \\ \frac{1}{2}(a_{21}+a_{12}) & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow ker(T) = \left\{ \begin{bmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{bmatrix}, \forall a_{12} \in \mathbb{R} \right\}$, which is a subspace of V

By simply identifying the output from the problem statement,

$\Rightarrow im(T) = \left\{ \begin{bmatrix} a_{11} & \frac{1}{2}(a_{21}+a_{12}) \\ \frac{1}{2}(a_{21}+a_{12}) & a_{22} \end{bmatrix}, \forall a_{11} \ldots a_{22} \in \mathbb{R} \right\}$, which is a subspace of V

$dim(ker(T)) = 1$

$dim(im(T)) = 3$

$\Rightarrow dim(T) = 4 = 1 + 3$

Prop: $ker(T)$ is a subspace of V

pf: Using the subspace test, $0 \in ker(T)$ since $T0 = 0$. If $v_1, v_2 \in ker(T)$ and $\alpha \in \mathbb{F}$, then $T(v_1 + \alpha v_2) = Tv_1 + T\alpha v_2 = 0 + \alpha 0 = 0$, so $ker(T)$ is closed under addition & scalar multiplication.

Prop: $im(T)$ is a subspace of V

pf: [This was left as an exercise to the reader]

Define: The **"KERNEL"** of T is: $null(T) \coloneqq dim(ker(T))$

Define: The **"RANK"** of T is: $rank(T) \coloneqq dim(im(T))$

Theorem: **"Rank-Nullity Theorem"**: If V is finite-dimensional, then $dim(V) = null(T) + rank(T)$

pf: Let $n := dim(V),\ k := null(T) = dim(ker(T)) \leq n$

Let $\{v_1, \ldots, v_k\}$ be a basis of $ker(T)$

Extend $\{v_1, \ldots, v_k\}$ to a basis $\{v_1, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_n\}$ of V

Now if $v \in V$ and $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$, then

$Tv = \alpha_{k+1} v_{k+1} + \ldots + \alpha_n v_n$ (all kernel terms drop out),

So $im(T) = span\{Tv_{k+1}, \ldots, Tv_n\}$

Moreover, if $\beta_{k+1} Tv_{k+1} + \ldots + \beta_n Tv_n = 0$, then

$\beta_{k+1} Tv_{k+1} + \ldots + \beta_n Tv_n \in ker(T)$

Hence, $\alpha_1 = \ldots = \alpha_k = -\beta_{k+1} = \ldots = -\beta_n = 0$ because $v_1, \ldots, v_n$ are a basis of V

This shows that $Tv_{k+1}, \ldots, Tv_n$ are linearly independent.

Therefore, $rank(T) = dim(im(T)) = n - k$

Prop: $im(T)$ is a subspace of W

[something to ponder on until the next section]

DISCUSSION 2 | *Thursday, 4/17*

Question:	If T is a linear map, must $T(0) = 0$?
Answer:	Yes!
Proposed pf:	[Attributed to Kaelan] $T(0) = T(\alpha v); \alpha = 0, v \in V$ $T(\alpha v) = \alpha T(v) = 0(T(v))$ $T(0) = 0(T(v)) = 0$

Guiding Question: What do the kernel and image have to do with injectivity and surjectivity?

Prop 1: If the kernel is not just $\{0\}$, then T is not injective

pf:

Assume $ker(T) \neq \{0\}$
$\Rightarrow |ker(T)| > 1$
$\Rightarrow \exists\, v \in ker(T)$ such that $v \neq 0$
$\Rightarrow T(v) = 0$
$\Rightarrow \therefore T$ is NOT injective as this violates the definition of injectivity

"If $ker(T) \neq 0$, then T is not injective" ⇔ "If T is injective, then $ker(T) = \{0\}$"

Prop 2: If $ker(T) = \{0\}$, then is T injective?

pf: [Attributed to Brandon]

Suppose $ker(T) = \{0\}$
Let $v_1, v_2 \in V$ such that $T(v_1) = T(v_2)$
We want to show that $v_1 = v_2$

$T(v_1) = T(v_2)$
$\Rightarrow T(v_1) - T(v_2) = 0$
$\Rightarrow v_1 - v_2 \in ker(T) = \{0\}$
$\Rightarrow v_1 - v_2 = 0$
$\Rightarrow v_1 = v_2$
$\Rightarrow T$ is injective

Prop 3:	For linear $T: V \rightarrow W$, T being surjective means $im(T) = W$
pf:	None given by instructor; “self-evident”

Parting Question: For $T: V \rightarrow W,\ V = \{v_1,\ v_2,\ \dots,\ v_n\},\ W = \{w_1,\ w_2,\ \dots,\ w_n\}$, what can we say about the following relationship?:

$\{v_1,\ v_2,\ \dots,\ v_n\}$ linearly independent $\overset{?}{\Longleftrightarrow}$ $\{T(v_1),\ T(v_2),\ \dots,\ T(v_n)\}$ linearly independent

[left by instructor as an exercise for the reader]

LECTURE 3 | *Friday, 4/18* [Alan]

EXERCISES

Suppose we have a linear map $T: V \rightarrow W$, where V and W are finite-dimensional vector spaces.

1. Show that $im(T)$ is a subspace of W.
    a. We also have that $ker(T)$ is a subspace of V.
2. Show that $ker(T) = \{0\}$ if and only if T is injective.
    a. T is injective if $\forall\, v, v' \in V, T(v) = T(v') \Rightarrow v = v'$.
    b. We also have that $im(T) = W$ if and only if T is surjective (by definition).
3. Show that if $dim(V) > dim(W)$, then T is not injective.

SOLUTIONS TO EXERCISES

Exercise 1 Solution

Suppose we have $T(u), T(\alpha v) \in im(T)$, where $u, v \in V$, $\alpha \in \mathbb{F}$. Then, since V is a vector space, we must have $u + \alpha v \in V \Rightarrow T(u + \alpha v) = T(u) + \alpha T(v) \in im(T)$. Thus, we have shown that $im(T)$ is closed under addition and scalar multiplication. Moreover, we have $T(0) = 0$, so $im(T)$ is non-empty. Therefore, $im(T)$ is a subspace of W.

Exercise 2 Solution

First, we show that $ker(T) = \{0\} \Rightarrow T$ is injective.

Suppose we have $ker(T) = \{0\}$, and $T(v) = T(v')$ for some $v, v' \in V$. Then, $T(v) - T(v') = T(v - v') = 0$. Thus, we must have $v - v' \in ker(T)$. However, sine $ker(T) = \{0\}$, we must have $v - v' = 0 \Rightarrow v = v'$. So, for all $v, v' \in V, T(v) = T(v') \Rightarrow v = v'$, i.e. T is injective.

Now, we show that T is injective $\Rightarrow ker(T) = \{0\}$.

If T is injective, then $T(v) = T(v') \Rightarrow v = v' \ \forall v, v' \in V$. Let $v' = 0$. Then, we have $T(v) = 0 \Rightarrow v = 0 \ \forall v \in V$. Since the only v satisfying $T(v) = 0$ is $v = 0$, we have that $ker(T) = \{0\}$.

Exercise 3 Solution

Since $im(T)$ is a subspace of W (**Exercise 1**), we have that

$rank(T) = dim(im(T)) \leq dim(W)$.

Combining this with the given inequality yields

$rank(T) \leq dim(W) < dim(V) \Rightarrow rank(T) < dim(V) \Rightarrow dim(V) - rank(T) > 0$.

By the rank-nullity theorem, $dim(V) - rank(T) = null(T) = dim(ker(T))$, so $dim(ker(T)) > 0$. Thus, $ker(T)$ cannot be $\{0\}$, which has dimension 0. Since T is injective only if $ker(T) = \{0\}$ (**Exercise 2**), we have that T is not injective.