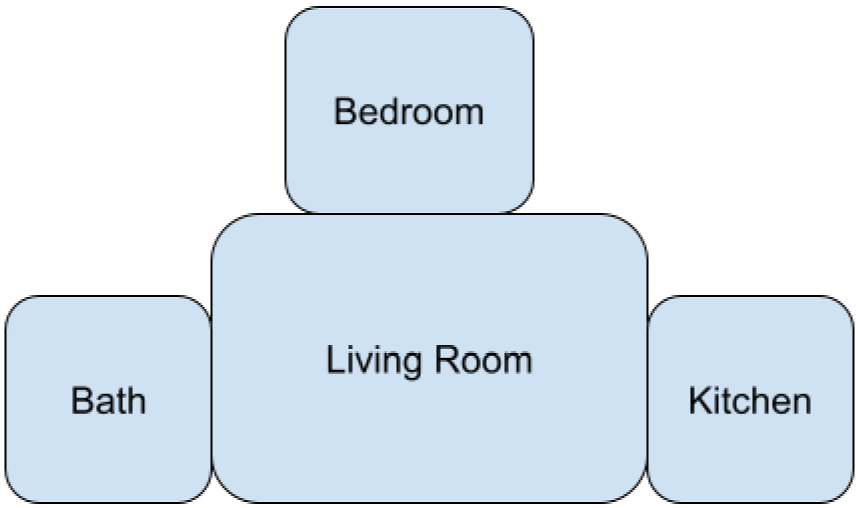
Lecture 1

Apartment Analogy

Problem: There is an apartment with four rooms. Each room has a light and a switch. Each switch toggles the light in its room and all adjacent rooms. The graph of the apartment follows:



Question:

1) Can I turn on only the living room lights?
2) Are all configurations possible?
3) Why is this linear algebra?

Class Consensus:
Every time you press a button it flips an even number of lights. So the number of lights that are on is always even (assuming we start from zero).

So only the living room light on is impossible.

Why is this linear Algebra?
We can model this with linear algebra.
Have a variable for each room's lighting state {0,1}. "1" means turned on. Pressing a button is like adding a vector.

There is a variable for each room's lightning state x_i. The whole system can be described with the set X.

Let's model this with a field we name F_2. The elements of the field can take on the values {0,1}. The addition table for this field is below (0+0=0, 1+1=0, 0+1=1, 1+0=1)

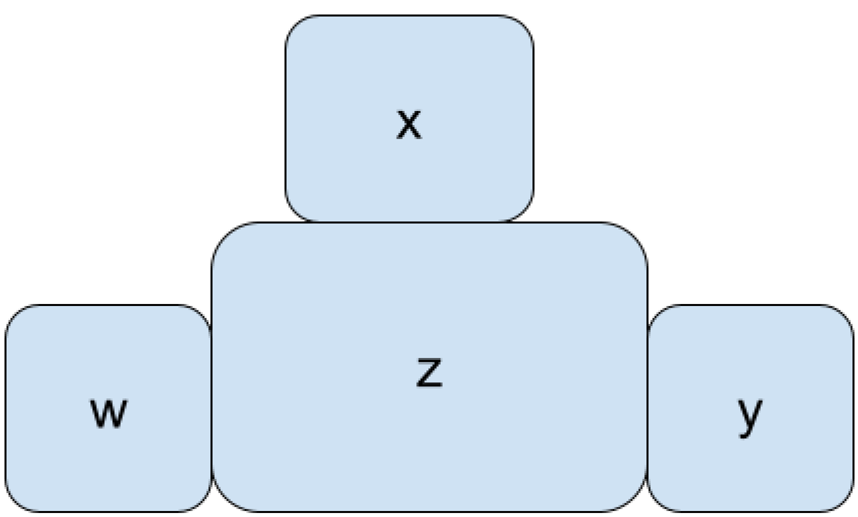
+	0	1
0	0	1
1	1	0

Student question: is this a valid field? Answer: you'll find out in the hw

Vector space:
Configurations of on/off lights

$F_2{}^4$ = {(w,x,y,z), w, x, y, z $\in F_2$
Each room is assigned one of these F_2 variables: shown below
Notation: z is the living room

- When we start, all lights are off: (0,0,0,0)
- When pressing a button, we essentially add a vector to the current system vector
    - pressing 'y' button is adding (0,0,1,1)
        - Let's call this vector b_y
    - b_w = (1, 0, 0, 1)
    - b_x = (0,1,0,1)
    - b_z = (1,1,1,1)
-



Question 2 above can be thought of as the following question: which elements of F_2^4 can be represented as a linear combination of (0,0,0,0) and each button vector? In other words, what is the **span** of the button vectors?
Definition → (Linear combination): given vectors $v_1, \dots, v_n \in V$, scalars $c_1, \dots, c_n \in F$. A linear combination of $v_1, \dots, v_n$ is $c_1 v_1 + \dots + c_n v_n$

Definitition → (span) set of all linear combinations span $(v_1, \dots, v_n) = \{c_1 v_1 + \dots + c_n v_n, c_1, \dots, c_n \in F\}$

So, to answer question 1, we need to show that (0,0,0,1) either is or is not in the span of the button vectors. To do this, we need to find if (0,0,0,1) is linearly independent of the button vectors.

Question to the class: Can we use vectors to prove (0,0,0,1) is impossible?

Find whether [0, 0, 0, 1] in the span

Span$\{b_x, b_Y, b_z, b_w\}$
$= \{ c_w \cdot b_w + c_x \cdot b_x + c_Y \cdot b_Y + c_z \cdot b_z \}$
$= \{ c_w \cdot [1, 1, 1, 1]^T + c_x \cdot [0, 1, 0, 1]^T + c_Y \cdot [0, 0, 1, 1]^T + c_z \cdot [0, 0, 0, 1]^T \}$
$= \{[c_z + c_w$
$c_z + c_x$
$c_z + c_Y$
$c_z + c_Y + c_x + c_w] : c_x, c_Y, c_z, c_w \in \mathbb{F}_2\}$

$c_z + c_w = 0$
$c_z + c_x = 0$
$c_z + c_Y = 0$
$c_z + c_Y + c_x + c_w = 1$

Solve for C's

An important note: We note that the last vector is a linear combination of the other vectors → it is redundant, or linearly dependent.

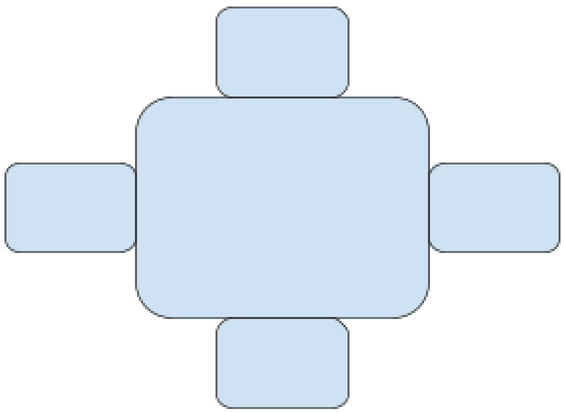
Mod2

Discussion 1

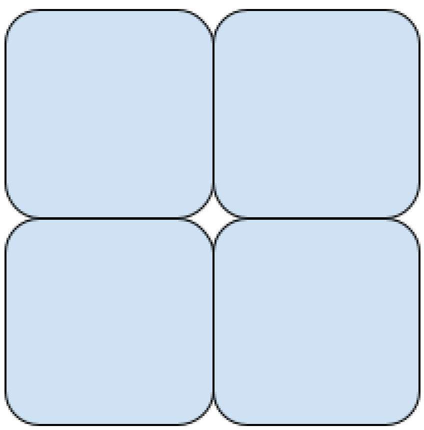
Let's expand on the lights example from yesterday.

Today's task:

1) Let us prove that the span of the example from yesterday is not equal to the field F2^4
2) Let us try two other apartment configurations.



a)



b)

Solutions

1) One way to find if the span is equal to the whole field is to find the kernel. If the kernel is non-zero, it means that the span is **not** equal to the whole field.
2)

Math 33a review

Question: how would I find the span($(3\ 1\ 4)^T$, $(2\ -\ 3\ 5)^T$, $(\ 5\ 9\ 2)^T$?

Example: let’s find the span of
V1 = (1,1,1,0,0)
V2 = (0,0,2,0,1)
V3 = (0,0,4,0,2)

Normally, we would place the vectors in columns and take the following approach:

We begin with the matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Step 1: Subtract row 1 from rows 2 and 3

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Step 2: Switch orders of rows 2 and 3

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Step 3: Subtract row 5 from row 3

Step 4: Subtract row 3 from row 5

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

However, this doesn't actually give us useful information about the span.

Consequently, we take a different approach: we place the vectors in the rows of a matrix rather than in the columns.

After row reducing, we get this matrix:
[1 1 0 0 -½]
[0 0 1 0 ½]
[0 0 0 0 0]

Therefore, the span is = span (1 1 0 0 -½) + (0 0 1 0 ½)

DEF (linear dependence)

A list of vectors $v_1, \ldots, v_n$ is linearly dependent if there exists $i \in \{1, \ldots, n\}$ such that $v_i \in \operatorname{span}(\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\})$

Lecture 2

Span & Linear Independence

Let V be a vector space over $\mathbb{F}$.

Definitions:

- **Span of** $S \subseteq V$:

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{k} \alpha_i v_i \mid k \in \mathbb{Z}_{\geq 0}, \alpha_i \in \mathbb{F}, v_i \in S \right\}$$

- **Linear Dependence**:
  A set $S \subseteq V$ is **linearly dependent** if

$$\exists k \in \mathbb{Z}_{\geq 0}, v_i \in S, \alpha_i \in \mathbb{F} \text{ such that } \sum \alpha_i v_i = 0,$$

  and the α_i are **not all 0** (i.e., a *nontrivial* linear combination).

Lemma (Linear Dependence):

Let $v_1, \ldots, v_k \in V$.
Then $S = \{v_1, \ldots, v_k\}$ is linearly dependent **if and only if**

$$v_j \in \operatorname{span}\{v_1, \ldots, v_{j-1}\} \text{ for some } j,$$

in which case

$$\operatorname{span}(S \setminus \{v_j\}) = \operatorname{span}(S)$$

Example:

Let $V = \mathbb{R}^2$

$$v_1 = (1, 0), \quad v_2 = (2, 0), \quad v_3 = (0, 1)$$

Since $v_2 = 2v_1$, we have:

$$v_2 \in \operatorname{span}\{v_1\}$$

Proof:

Suppose S is linearly dependent so that

$$\sum_{i=1}^{k} \alpha_i v_i = 0$$

for some $\alpha_i \in \mathbb{F}$, not all zero.

Let j be the **largest** index such that $\alpha_j \neq 0$. Then:

$$v_j = \sum_{i=1}^{j-1} \left(\frac{-\alpha_i}{\alpha_j} \right) v_i \quad \Rightarrow \quad v_j \in \operatorname{span}\{v_1, \ldots, v_{j-1}\}$$

Now consider the opposite direction:

$$v_j = \sum_{i=1}^{j-1} \alpha_i v_i$$

$$v_j - \sum_{i=1}^{j-1} \alpha_i v_i = 0$$

$$\sum_{i=1}^{j-1} \alpha_i v_i + (-1) v_j = 0$$

This is a **nontrivial linear combination**, so:

$$\sum_{i=1}^{j} \alpha_i v_i = 0$$

Clearly,

$$\text{span}(S \setminus \{v_j\}) \subseteq \text{span}(S)$$

Also, $v = \beta_1 v_1 + \cdots + \beta_k v_k$, and

$$v_j = \beta_1 v_1 + \cdots + \beta_{j-1} v_{j-1} + \beta_{j+1} v_{j+1} + \cdots + \beta_k v_k + \beta_j v_j$$

Then

$$v - \beta_j v_j = \beta_1 v_1 + \cdots + \beta_{j-1} v_{j-1} + \beta_{j+1} v_{j+1} + \cdots + \beta_k v_k \Rightarrow v = \beta_j v_j + (\text{something in span}(S \setminus \{v_j\}))$$

So

$$\text{span}(S) \subseteq \text{span}(S \setminus \{v_j\}) \Rightarrow \text{span}(S) = \text{span}(S \setminus \{v_j\})$$

Lemma (Steinitz exchange):
If $U = \{u_1, \ldots, u_m\}$ is linearly independent in V and
$W = \{w_1, \ldots, w_n\}$ spans V, then $m \leq n$ and

$$\exists v_1, \ldots, v_{n-m} \in W \text{ such that } U \cup \{v_1, \ldots, v_{n-m}\} \text{ spans } V$$

(*exchange m in, m out*)

Pf. Fix n. We will prove the lemma by induction on m. This means proving for $m = 0$, then proving that if it holds for some m, then it must hold for $m + 1$.

Suppose $m = 0$. Then $U = \emptyset$,

$$W = \{w_1, \ldots, w_n\}$$

Indeed $0 \leq n$ and W spans V.

Now assume the lemma holds for some m

and suppose $U = \{u_1, \ldots, u_{m+1}\}$ is linearly independent
and $W = \{w_1, \ldots, w_n\}$ spans V

Since $\{u_1, \ldots, u_m\}$ is linearly independent, by the inductive assumption $m \leq n$ and

$$\exists v_1, \ldots, v_{n-m} \in W \text{ s.t. } \{u_1, \ldots, u_m, v_1, \ldots, v_{n-m}\} \text{ spans } V$$

Hence

$$u_{m+1} = \alpha_1 u_1 + \cdots + \alpha_m u_m + \beta_1 v_1 + \cdots + \beta_{n-m} v_{n-m}$$

for some α's and β's

We must have some terms with β's because otherwise U would not be linearly independent, i.e.,

$$n - m \geq 1 \Rightarrow m + 1 \leq n$$

Discussion 2

Review

Yesterday we did the Linear Dependence lemma and the Steinitz exchange

Linear Dependence lemma

If v_1,...v_n are linearly dependent, then there exists a vector v_j in the v_1…v_n such that v_j \in span(v_1,...v_n) and, span(v_1,...v_n) = span(v_1,...,v_n / {v_j})

Steinitz exchange

If U is a linearly independent set, W is a spanning set, both finite, then the cardinality of U <= cardinality of W, and there exists a subset of W' of W such that U union W' is spanning.

So what are we getting at here?
Essential Question: What is a **basis**?

Def(basis) = set of vectors for a given vector space that are **spanning and independent**

Eg in R^3 we have (0,0,1) (0,1,0) and (1,0,0) which are our "basic directions of movement"

Another important concept: **Dimension** number of basis vectors in a space

Question: do all bases have the same size?

Proof by contradiction

- Steinitz Exchange
- Have U and W then we can say one is lin ind, one is spanning
  - Do this both ways- meaning U≤W and W≤U so U=W

Question: If S is maximally independent, ie if adding any vector to S makes it dependent does S span everything?

- Proof outline: consider adding a vector that is not included in the span to S, but doing so would contradict the assumption that S is maximally independent.

Proof:

Suppose S is a maximally independent set in a vector space V, but does not span V.
Then there exists a vector $v \in V$ such that $v \notin \text{span}(S)$.

Now consider $S' = S \cup \{v\}$.
Since $v \notin \text{span}(S)$, S' is still linearly independent.

This contradicts the assumption that S is maximally independent (i.e., adding any vector makes it dependent).

Therefore, S must span V.

$$\boxed{S \text{ spans } V}$$

Question: If S is minimally spanning, is S independent?

- Proof outline: use linear dependence lemma to show that if it was not independent, you would be able to remove a vector and still it would span the same set, but that contradicts the given that S is minimally spanning.

More formally:

Proof:

Suppose S is not linearly independent.
Then by the linear dependence lemma, there exists a vector $v \in S$ such that

$$\operatorname{span}(S \setminus \{v\}) = \operatorname{span}(S)$$

This means removing v still leaves a spanning set, contradicting the assumption that S is minimally spanning.

Therefore, S must be linearly independent.

$$\boxed{S \text{ is linearly independent}}$$

Lecture 3

Review of Steinitz exchange

Dimensionality

Definition:
A set $B \subseteq V$ is a basis of V
if it spans V and is linearly independent in V.

The number of elements in B is called the dimension of V.

Example:
Let $V = \mathbb{R}^3$,

$$B = \{v_1, v_2, v_3\} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$$

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3\} = \text{span}\{v_1, v_2, v_3\} \quad \Rightarrow \text{spans } V$$

Also,

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \vec{0} = (0,0,0) \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \quad \Rightarrow \text{linearly independent in } V$$

Therefore,

$$\dim(V) = 3$$

Practice problems

What is the dimension of V for:

1) V = R^2x2 = {[a1, a2],[a3,a4]}:ai \in R
    a) Strategy for this problem: find a basis
    b) Basis was of dimension 4 → one matrix with a_{11} = 1, one with a_{12} = 1, one with a_{21} = 1, one with a_{22} = 1
2) V = P_n(R) = {polynomial of <= nth degree}
    a) Basis was just a polynomial of n degree with constants = 1
       $1 + x + x^2 \ldots + x^n$
    b) This is of dimension n + 1

More Review for Thursday's content

Let V be a vector space with basis B, where $\#B = n$
($\#$ = number of elements)

(a) If S spans V, then $\#S \geq n$

(b) If S is linearly independent in V, then $\#S \leq n$

(c) If S is a basis of V, then $\#S = n$

Proof (a): By Steinitz Exchange Lemma,

$$\#S \geq \#B = n \quad \text{because } B \text{ is linearly independent}$$

Proof (b): By Steinitz Exchange Lemma,

$$\#S \leq \#B = n \quad \text{because } B \text{ spans}$$

Proof (c): S satisfies the hypotheses of (a) and (b),
so $\#S = n$