Fractal uncertainty principles for ellipsephic sets

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The fractal uncertainty principle (Dyatlov–Zahl, 2016)

“No function can be localized in both position and frequency close to a fractal set.”

- Applications to quantum chaos (eigenfunction control and spectral gaps on hyperbolic surfaces).
- Connections to harmonic analysis (additive energy, Fourier decay, and Fourier restriction estimates; additive combinatorics; spectral sets).
Continuous uncertainty principles

Let $F_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ ($0 < h \ll 1$) be the unitary semiclassical Fourier transform

$$F_h f(\xi) := \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-ix\xi/h} f(x) \, dx.$$  

Continuous uncertainty principles (Dyatlov–Zahl, 2016)

An $h$-dependent family of sets $\{X_h\}_{h>0} \subseteq \mathcal{P}(\mathbb{R})$ is said to satisfy an uncertainty principle with exponent $\beta \in \mathbb{R}$ if

$$\| \mathbf{1}_{X_h} F_h \mathbf{1}_{X_h} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = O(h^\beta) \quad \text{as } h \to 0.$$  

(The subscript on $X$ is typically elided.)
Example: $X = [0, h]$.

By Hölder’s inequality,

$$
\| 1_X \mathcal{F}_h 1_X \|_{L^2 \to L^2} \leq \| 1_{[0,h]} \|_{L^\infty \to L^2} \| \mathcal{F}_h \|_{L^1 \to L^\infty} \| 1_{[0,h]} \|_{L^2 \to L^1}
\]

$$
$$
= h^{1/2} \cdot (2\pi h)^{-1/2} \cdot h^{1/2},
$$

so $X$ satisfies an uncertainty principle with exponent $\frac{1}{2}$. 
Continuous fractal uncertainty principles

For “regular” fractal sets $X \subseteq [0, 1]$ of “dimension” $\delta \in [0, 1]$, we have the basic fractal uncertainty principle (FUP) exponent

$$\beta_0 := \max \left\{ 0, \frac{1}{2} - \delta \right\}.$$

Can this be improved upon (by obtaining $\beta > \beta_0$ for $\delta$-regular families of sets)?

- Yes – when $\delta < 1$, we can obtain $\beta > 0$: improvement for $\delta \geq \frac{1}{2}$ (Bourgain–Dyatlov, 2017).
- Yes – when $\delta > 0$, we can obtain $\beta > \frac{1}{2} - \delta$: improvement for $\delta \leq \frac{1}{2}$ (Dyatlov–Jin, 2018).
An **ellipsephic** ([ˌɪlpˈsɛf.ɪk]) set in base $M$ is a set consisting of all $k$-digit integers in base $M$ with digits in some nonempty **alphabet** $\mathcal{A} \subseteq \mathbb{Z}_M := \{0, 1, \ldots, M - 1\}$. Such a set is denoted $C_k(M, \mathcal{A})$ (or simply $C_k$). In other words,

$$C_k = C_k(M, \mathcal{A}) := \left\{ \sum_{d=0}^{k-1} a_d M^d : a_d \in \mathcal{A} \right\}.$$ 

Note that $C_k \subseteq \mathbb{Z}_N$ for $N := M^k$ and $|C_k| = |\mathcal{A}|^k = N \log_M |\mathcal{A}|$.

The **dimension** of $C_k(M, \mathcal{A})$ is $\delta := \log_M |\mathcal{A}| \in [0, 1]$. We will not consider trivial alphabets with $\delta = 0$ ($|\mathcal{A}| = 1$) or $\delta = 1$ ($|\mathcal{A}| = M$).
Example: $M = 10, \mathcal{A} = \{2, 7\}$.

$$C_2(M, \mathcal{A}) = \{22, 27, 72, 77\}$$
$$\delta = \log_{10} 2 \approx 0.3$$
Let $\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N$ be the unitary discrete Fourier transform

$$\mathcal{F}_N u(j) := \frac{1}{\sqrt{N}} \sum_{\ell \in \mathbb{Z}_N} e^{-2\pi ij\ell/N} u(\ell) = \frac{1}{\sqrt{N}} \sum_{\ell \in \mathbb{Z}_N} \omega_N^{j\ell} u(\ell).$$

Discrete fractal uncertainty principles (Dyatlov–Jin, 2017)

A family of ellipsephic sets $\{C_k(M, A)\}_{k \geq 1}$ is said to satisfy an uncertainty principle with exponent $\beta \in \mathbb{R}$ if

$$\|1_{C_k} \mathcal{F}_N 1_{C_k} \|_{\ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)} \lesssim M, A \ N^{-\beta}.$$
Example: $M = 10$, $A = \{2, 7\}$, $k = 1$.

$$\|1_{C_k} F_N 1_{C_k}\|_2 = \|1_{\{2,7\}} F_{10} 1_{\{2,7\}}\|_2 = \left\| \frac{1}{\sqrt{10}} \begin{bmatrix} \omega_{10}^{2.2} & \omega_{10}^{2.7} \\ \omega_{10}^{7.2} & \omega_{10}^{7.7} \end{bmatrix} \right\|_2$$

Example: $M = 10$, $A = \{0, 5\}$, $k = 1$.

$$\|1_{C_k} F_N 1_{C_k}\|_2 = \left\| \frac{1}{\sqrt{10}} \begin{bmatrix} \omega_{10}^{0.0} & \omega_{10}^{0.5} \\ \omega_{10}^{5.0} & \omega_{10}^{5.5} \end{bmatrix} \right\|_2 = \left\| \frac{1}{\sqrt{10}} \begin{bmatrix} \omega_{10}^{2.2} & \omega_{10}^{2.7} \\ \omega_{10}^{7.2} & \omega_{10}^{7.7} \end{bmatrix} \right\|_2$$

Notice that $\{0, 5\} + 2 = \{2, 7\}$ and

$$\begin{bmatrix} \omega_{10}^{0.2} \\ \omega_{10}^{5.2} \end{bmatrix} \begin{bmatrix} \omega_{10}^{0.0} & \omega_{10}^{0.5} \\ \omega_{10}^{5.0} & \omega_{10}^{5.5} \end{bmatrix} \begin{bmatrix} \omega_{10}^{2.0} \\ \omega_{10}^{2.5} \end{bmatrix} \begin{bmatrix} \omega_{10}^{2.2} \\ \omega_{10}^{2.7} \end{bmatrix} = \begin{bmatrix} \omega_{10}^{2.2} & \omega_{10}^{2.7} \\ \omega_{10}^{7.2} & \omega_{10}^{7.7} \end{bmatrix}.$$
For ellipsephic sets of dimension $\delta \in [0, 1]$, we have the basic FUP exponent

$$\beta_0 := \max \left\{ 0, \frac{1}{2} - \delta \right\}.$$ 

Can this be improved upon (by obtaining $\beta > \beta_0$ for ellipsephic sets of dimension $\delta$)?

- Yes – for all $0 < \delta < 1$, we can obtain $\beta > \beta_0$ (Dyatlov–Jin, 2017).
Discrete fractal uncertainty principles

Proof (basic FUP exponent):

\[
\| 1_{C_k} \mathcal{F}_N 1_{C_k} \|_2 \leq \| \mathcal{F}_N \|_2 = 1 = N^{-0}
\]

\[
\| 1_{C_k} \mathcal{F}_N 1_{C_k} \|_2 \leq \| 1_{C_k} \mathcal{F}_N 1_{C_k} \|_F = \sqrt{|C_k|^2 \left( \frac{1}{\sqrt{N}} \right)^2} = N^{-\left( \frac{1}{2} - \delta \right)}
\]
Let $r_k = r_k(M, A) := \| \mathbb{1}_{C_k(M, A)} \mathcal{F}_N \mathbb{1}_{C_k(M, A)} \|_2$.

- **Upper bound:**
  $$\beta \leq \frac{1 - \delta}{2}.$$

- **Apply** $\mathbb{1}_{C_k} \mathcal{F}_N \mathbb{1}_{C_k}$ to $1\{x\}$ for some $x \in C_k$.

- **Alphabet shift:** if $a \in \mathbb{Z}_M$ and $A \subseteq \{0, 1, \ldots, (M - 1) - a\}$, then
  $$r_k(M, A + a) = r_k(M, A).$$

- **Notice that** $C_k(M, A + a) = C_k(M, A) + (a \cdots a)_M$ and apply the shift theorem for the DFT.

- **Submultiplicativity:**
  $$r_{k_1 + k_2} \leq r_{k_1} r_{k_2}.$$ 

- **Notice that** $C_{k_1 + k_2} = C_{k_1} C_{k_2}$ (in the sense of *concatenation*) and use an FFT-like decomposition.
Let \( \beta_k = -\log_N r_k = -\frac{\log_M r_k}{k} \).

Fekete’s lemma applied to the subadditive sequence \( \{\log_M r_k\}_{k \geq 1} \) allows us to compute the maximal \( \beta \) as

\[
\beta = \lim_{k \to \infty} \beta_k = \sup_{k \geq 1} \beta_k.
\]
Recent work

**How does (the maximal) $\beta$ depend on $(M, A)$?**

![Graph showing the approximation of FUP exponents](image)

**Figure:** Numerically approximated FUP exponents for all alphabets with $M \leq 10$. 
For any $\delta \leq \frac{1}{2}$, the improvement over the basic exponent can be arbitrarily small, in that there exist sequences $\{(M_j, A_j)\}$ with $\delta(M_j, A_j) \to \delta$ and $\beta(M_j, A_j) \to \beta_0$ (Dyatlov–Jin, 2017).

Is this also true for $\delta > \frac{1}{2}$? (Dyatlov, 2019)

- Yes (·, 2021).
- For some sequences, the improvement over the basic exponent might even be (nearly) exponentially small. (We have an upper bound for $\beta_1$ so far.)
Which bases/alphabets attain the upper bound $\beta = \frac{1-\delta}{2}$? (Dyatlov–Jin, 2017)

- Numerical experiments ($M \leq 25$; later, $M \leq 39$) suggest that these might be the only ones.
Thank you for your attention!