Deconvolutional determination of the nonlinearity in a semilinear wave equation

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Introduction
Consider the semilinear wave equation

\[
\begin{cases}
(\partial_{tt} - \Delta_x)u(t, x) = F(u(t, x)), & (t, x) \in \mathbb{R} \times \mathbb{R}^3; \\
u(0, \cdot) = u_0; \\
\partial_t u(0, \cdot) = u_1.
\end{cases}
\]

When the initial data \((u_0, u_1)\) is “small”, the solution to this equation will “behave in the distant future or past” like the solution to a linear wave equation

\[
\begin{cases}
(\partial_{tt} - \Delta_x)u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^3; \\
u(0, \cdot) = u_0^*; \\
\partial_t u(0, \cdot) = u_1^*.
\end{cases}
\]

This phenomenon is called “(small-data) scattering”.

A commonly asked question in the study of nonlinear dispersive PDEs (e.g., NLS, NLW, Klein–Gordon) is:

“Is the nonlinearity determined by how it scatters solutions?”
Strong assumptions on the nonlinearity (e.g., analyticity) are often made to obtain a positive answer.

(Sá Barreto–Uhlmann–Wang, 2020): quintic-type nonlinearities ($|F(u)| \approx |u|^5$) for the NLW equation in 3D; complicated argument with many assumptions on the nonlinearity.

(Killip–Murphy–Vişan, 2023): power-type nonlinearities for the NLS equation in 2D; much simpler argument with few assumptions on the nonlinearity!

We adapt the techniques of Killip, Murphy, and Vişan to the setting considered by Sá Barreto, Uhlmann, and Wang.
Definition (Admissible nonlinearity)

A nonlinearity $F : \mathbb{R} \to \mathbb{R}$ is considered **admissible** if:

1. $F(0) = 0$
2. $|F(u) - F(v)| \lesssim (|u|^4 + |v|^4)|u - v|$ (so $|F(u)| \lesssim |u|^5$)
3. $F(-u) = -F(u)$

The archetypal admissible nonlinearity is $F(u) = \pm |u|^4 u$. The corresponding equation is known as the **defocusing/focusing energy-critical** NLW equation because the rescaling $u(t, x) \mapsto \lambda u(\lambda^2 t, \lambda^2 x)$ (for $\lambda > 0$) preserves the **energy** of solutions,

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 + |\partial_t u(t, x)|^2 \, dx \pm \frac{1}{6} \int_{\mathbb{R}^3} |u(t, x)|^6 \, dx.$$
The NLW equation can be written as

$$\partial_t \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F(u(t)) \end{bmatrix},$$

so

$$e^{-At} \partial_t \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = e^{-At} \mathcal{A} \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} + e^{-At} \begin{bmatrix} 0 \\ F(u(t)) \end{bmatrix}.$$ 

Hence

$$\partial_t \left( e^{-At} \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} \right) = e^{-At} \begin{bmatrix} 0 \\ F(u(t)) \end{bmatrix}.$$ 

Integrating and rearranging, we obtain

$$\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = e^{At} \begin{bmatrix} u(0) \\ \partial_t u(0) \end{bmatrix} + \int_0^t e^{A(t-s)} \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} \, ds.$$
The **propagator** for the linear wave equation is

$$
U(t) := e^{At} = \exp \begin{bmatrix} 0 & t \\ t\Delta & 0 \end{bmatrix} = \begin{bmatrix} \cos(t|\nabla|) & \frac{\sin(t|\nabla|)}{|\nabla|} \\
-|\nabla| \sin(t|\nabla|) & \cos(t|\nabla|) \end{bmatrix},
$$

where $|\nabla| = \sqrt{-\Delta}$.

We therefore have the **Duhamel formula**

$$
\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = U(t) \begin{bmatrix} u(0) \\ \partial_t u(0) \end{bmatrix} + \int_0^t U(t-s) \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} \, ds.
$$
Introduction: solutions of the NLW equation

Definition (Solution)

A function \( u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \) is said to be a \textbf{(strong) global solution} of the NLW equation if \((u, \partial_t u) \in C^0_t \dot{H}^1_x(K \times \mathbb{R}^3) \times C^0_t L^2_x(K \times \mathbb{R}^3)\) and \(u \in L^5_t L_{x}^{10}(K \times \mathbb{R}^3)\) for all compact sets \(K \subseteq \mathbb{R}\) and if \(u\) satisfies the Duhamel formula

\[
\begin{bmatrix}
  u(t) \\
  \partial_t u(t)
\end{bmatrix} = \mathcal{U}(t) \begin{bmatrix}
  u_0 \\
  u_1
\end{bmatrix} + \int_0^t \mathcal{U}(t-s) \begin{bmatrix}
  0 \\
  F(u(s))
\end{bmatrix} ds.
\]

Theorem (Strichartz estimates)

If \(u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}\) is a global solution of the NLW equation, then

\[
\| (u, \partial_t u) \|_{L^\infty_t \dot{H}^1_x \times L^\infty_t L^2_x} + \| u \|_{L^5_t L_{x}^{10}} \lesssim \| (u_0, u_1) \|_{\dot{H}^1 \times L^2} + \| F(u) \|_{L^1_t L^2_x}.
\]
Theorem (Small-data scattering)

Let $F$ be an admissible nonlinearity for the NLW equation. Then there exists an $\eta > 0$ such that the NLW equation has a unique global solution $u$ satisfying

$$
\|(u, \partial_t u)\|_{L^\infty_t \dot{H}^1_x \times L^\infty_t L^2_x} + \|u\|_{L^5_t L^{10}_x} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}
$$

whenever $(u_0, u_1) \in B_\eta$, where

$$
B_\eta := \{(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) : \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} < \eta\}.
$$

(Continued on the next slide.)
Introduction: small-data scattering

Theorem (Small-data scattering, continued)

This solution **scatters in** $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ as $t \to \pm \infty$, meaning that there exist (necessarily unique) asymptotic states $(u_0^\pm, u_1^\pm) \in \dot{H}^1 \times L^2$ for which

$$\left\| \left[ \begin{array}{c} u(t) \\ \partial_t u(t) \end{array} \right] - \mathcal{U}(t) \left[ \begin{array}{c} u_0^\pm \\ u_1^\pm \end{array} \right] \right\|_{\dot{H}^1 \times L^2} \to 0 \quad \text{as} \quad t \to \pm \infty.$$ 

In addition, for all $(u_0^-, u_1^-) \in B_\eta$, there exists a unique global solution $u$ to the NLW equation and a unique asymptotic state $(u_0^+, u_1^+) \in \dot{H}^1 \times L^2$ for which the above holds.
The map

\[(u_0, u_1) \mapsto (u_0^+, u_1^+)\]

implicitly defined by this theorem (on some open ball \(B_\eta \subseteq \dot{H}^1 \times L^2\)) will be referred to as the **wave operator** and will be denoted \(W_F\).

The map

\[(u_0^-, u_1^-) \mapsto (u_0^+, u_1^+)\]

is known as the **scattering operator** and will be denoted \(S_F\).
Introduction: determination of the nonlinearity

Theorem (Determination of the nonlinearity)

Suppose that $F$ and $\tilde{F}$ are admissible nonlinearities for the NLW equation and that $B_\eta$ and $B_{\tilde{\eta}}$ are corresponding balls given by the small-data scattering theorem. If $W_F$ and $W_{\tilde{F}}$, or $S_F$ and $S_{\tilde{F}}$, agree on $B_\eta \cap B_{\tilde{\eta}}$ (that is, the smaller of the two balls), then $F = \tilde{F}$.

(We will only discuss the case where the wave operators agree as the case where the scattering operators agree can be treated similarly.)
Proof (outline)

- Small-data scattering and asymptotics for the wave (and scattering) operators

\[ W_F = \text{[formula]} = \text{[approximate formula]} + \text{[error]} \]

- Reduction to a convolution equation

\[ W_F = W_{\tilde{F}} \implies H \ast w = \tilde{H} \ast w \]

- Deconvolutional determination of the nonlinearity

\[ H \ast w = \tilde{H} \ast w \implies H = \tilde{H} \implies F = \tilde{F} \]
Small-data scattering
Theorem (Small-data scattering)

Let $F$ be an admissible nonlinearity for the NLW equation. Then there exists an $\eta > 0$ such that the NLW equation has a unique global solution $u$ satisfying

$$
\|(u, \partial_t u)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u\|_{L_t^5 L_x^{10}} \lesssim \|(u_0, u_1)\|_{\dot{H}_x^1 \times L_x^2}
$$

whenever $(u_0, u_1) \in B_\eta$, where

$$
B_\eta := \{(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) : \|(u_0, u_1)\|_{\dot{H}_x^1 \times L_x^2} < \eta\}.
$$
Proof (sketch)

Consider the nonempty complete metric space \((X, d)\), where

\[
X := \{u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} : (u, \partial_t u) \in C^0_t \dot{H}^1_x \times C^0_t L^2_x, u \in L^5_t L^{10}_x, \\
\| (u, \partial_t u) \|_{L^\infty_t \dot{H}^1_x \times L^\infty_t L^2_x} + \| u \|_{L^5_t L^{10}_x} \leq 2C \| (u_0, u_1) \|_{\dot{H}^1 \times L^2} \}
\]

for some constant \(C > 0\) and

\[
d(u, v) := \| (u, \partial_t u) - (v, \partial_t v) \|_{L^\infty_t \dot{H}^1_x \times L^\infty_t L^2_x} + \| u - v \|_{L^5_t L^{10}_x}.
\]

Define a map \(\Phi\) on \(X\) using the Duhamel formula,

\[
\begin{bmatrix}
(\Phi(u))(t) \\
(\partial_t \Phi(u))(t)
\end{bmatrix} := \mathcal{U}(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t \mathcal{U}(t - s) \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} \, ds.
\]
Proof (sketch)

Show that $\Phi$ is a contraction on $(X, d)$ whenever $(u_0, u_1) \in B_\eta$ and $\eta$ is sufficiently small. By the Banach fixed point theorem, we then have

$$
\begin{bmatrix}
  u(t) \\
  \partial_t u(t)
\end{bmatrix}
= 
\begin{bmatrix}
  (\Phi(u))(t) \\
  (\partial_t \Phi(u))(t)
\end{bmatrix}
= 
\mathcal{U}(t) 
\begin{bmatrix}
  u_0 \\
  u_1
\end{bmatrix}
+ 
\int_0^t \mathcal{U}(t - s) 
\begin{bmatrix}
  0 \\
  F(u(s))
\end{bmatrix}
ds.
$$

for some unique $u \in X$, meaning that $u$ is a solution!
This solution scatters in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ as $t \to \pm\infty$, meaning that there exist (necessarily unique) asymptotic states $(u_0^\pm, u_1^\pm) \in \dot{H}^1 \times L^2$ for which

$$
\left\| \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} - \mathcal{U}(t) \begin{bmatrix} u_0^\pm \\ u_1^\pm \end{bmatrix} \right\|_{\dot{H}^1 \times L^2} \to 0 \quad \text{as } t \to \pm\infty.
$$

In addition, for all $(u_0^-, u_1^-) \in B_\eta$, there exists a unique global solution $u$ to the NLW equation and a unique asymptotic state $(u_0^+, u_1^+) \in \dot{H}^1 \times L^2$ for which the above holds.
Small-data scattering

**Proof (sketch)**

WLOG, consider $t \to +\infty$. We want to show that

$$(u(t), \partial_t u(t)) \approx \mathcal{U}(t)(u_0^+, u_1^+) \quad \text{in } \dot{H}^1 \times L^2 \text{ as } t \to +\infty.$$ 

Since $\mathcal{U}(t)$ is unitary on $\dot{H}^1 \times L^2$ for all $t$ and $\mathcal{U}(t)^{-1} = \mathcal{U}(-t)$, this is equivalent to

$$\mathcal{U}(-t)(u(t), \partial_t u(t)) \approx (u_0^+, u_1^+) \quad \text{in } \dot{H}^1 \times L^2 \text{ as } t \to +\infty.$$ 

We found that

$$
\begin{bmatrix}
    u(t) \\
    \partial_t u(t)
\end{bmatrix} = \mathcal{U}(t) 
\begin{bmatrix}
    u_0 \\
    u_1
\end{bmatrix} + \int_0^t \mathcal{U}(t-s) 
\begin{bmatrix}
    0 \\
    F(u(s))
\end{bmatrix} \, ds,
$$

so we expect (and indeed, it can be shown) that

$$
\begin{bmatrix}
    u_0 \\
    u_1
\end{bmatrix} + \int_0^\infty \mathcal{U}(-s) 
\begin{bmatrix}
    0 \\
    F(u(s))
\end{bmatrix} \, ds = 
\begin{bmatrix}
    u_0^+ \\
    u_1^+
\end{bmatrix}.
$$
This argument shows that the wave operator is given by

\[ W_F \left( \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right) = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^\infty U(-t) \begin{bmatrix} 0 \\ F(u(t)) \end{bmatrix} \, dt, \]

where \( u \) is the solution of the NLW equation with initial data \( (u_0, u_1) \).

The **Born approximation** to \( W_F \) is

\[ W_F \left( \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right) \approx \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^\infty U(-t) \begin{bmatrix} 0 \\ F(u_{\text{lin}}(t)) \end{bmatrix} \, dt, \]

where

\[ \begin{bmatrix} u_{\text{lin}}(t) \\ \partial_t u_{\text{lin}}(t) \end{bmatrix} := U(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}. \]
Corollary (Small-data asymptotics for the wave operator)

Suppose that $F$ is an admissible nonlinearity for the NLW equation and that $B_\eta$ is a corresponding ball given by the small-data scattering theorem. If $u_{\text{lin}}$ denotes the solution of the linear wave equation with initial data $(u_0, u_1) \in B_\eta$, then (in $\dot{H}^1 \times L^2$) we have

$$W_F \left( \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right) = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ F(u_{\text{lin}}(t)) \end{bmatrix} \, dt + \mathcal{O} \left( \left\| \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right\| \cdot \dot{H}^1 \times L^2 \right).$$
Small-data scattering

Proof

Comparing the formula for the wave operator to that of its Born approximation, we see that we need to prove that

$$\left\| \int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ F(u(t)) - F(u_{\text{lin}}(t)) \end{bmatrix} \right\|_{\dot{H}^1 \times L^2} \lesssim \left\| \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right\|_{\dot{H}^1 \times L^2}^9,$$

which we will do by duality.
Small-data scattering

Proof

Fix some \((v_0, v_1) \in \dot{H}^1 \times L^2\) and let \(v_{\text{lin}}\) denote the solution of the linear wave equation with initial data \((v_0, v_1)\). Then

\[
\left\langle \int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ F(u(t)) - F(u_{\text{lin}}(t)) \end{bmatrix} dt, \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2} \]

\[
= \int_0^\infty \left\langle \begin{bmatrix} 0 \\ F(u(t)) - F(u_{\text{lin}}(t)) \end{bmatrix}, \mathcal{U}(t) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2} dt
\]

\[
= \int_0^\infty \left\langle \begin{bmatrix} 0 \\ F(u(t)) - F(u_{\text{lin}}(t)) \end{bmatrix}, \begin{bmatrix} v_{\text{lin}}(t) \\ \partial_t v_{\text{lin}}(t) \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2} dt
\]

\[
= \int_0^\infty \left\langle F(u(t)) - F(u_{\text{lin}}(t)), \partial_t v_{\text{lin}}(t) \right\rangle_{L^2} dt.
\]
By Hölder’s inequality, the properties of $F$, the estimates for $u$, and the Strichartz estimates, we have

\[
\left| \int_0^\infty \left\langle F(u(t)) - F(u_{\text{lin}}(t)), \partial_t v_{\text{lin}}(t) \right\rangle_{L^2} dt \right| \\
\leq \| F(u) - F(u_{\text{lin}}) \|_{L^1_t L^2_x} \cdot \| \partial_t v_{\text{lin}} \|_{L^\infty_t L^2_x} \\
\lesssim (\| u \|_{L^5_t L^{10}_x}^4 + \| u_{\text{lin}} \|_{L^5_t L^{10}_x}^4) \| u - u_{\text{lin}} \|_{L^5_t L^{10}_x} \cdot \| \partial_t v_{\text{lin}} \|_{L^\infty_t L^2_x},
\]

where

\[
\| u \|_{L^5_t L^{10}_x}^4 \lesssim \| (u_0, u_1) \|_{\dot{H}^1_x \times L^2_x}^4,
\]
\[
\| u_{\text{lin}} \|_{L^5_t L^{10}_x}^4 \lesssim \| (u_0, u_1) \|_{\dot{H}^1_x \times L^2_x}^4,
\]
\[
\| u - u_{\text{lin}} \|_{L^5_t L^{10}_x} \lesssim \| F(u) \|_{L^1_t L^2_x} \lesssim \| u \|_{L^5_t L^{10}_x}^5 \lesssim \| (u_0, u_1) \|_{\dot{H}^1_x \times L^2_x}^5,
\]
\[
\| \partial_t v_{\text{lin}} \|_{L^\infty_t L^2_x} \lesssim \| (v_{\text{lin}}, \partial_t v_{\text{lin}}) \|_{\dot{H}_x^1 \times L^2_x} \|_{L^\infty_t} = \| (v_0, v_1) \|_{\dot{H}^1_x \times L^2_x}. 
\]
Reduction to a convolution equation
Proposition (Reduction to a convolution equation)

Suppose that $F$ and $\tilde{F}$ are admissible nonlinearities for the NLW equation. For $\tau \in \mathbb{R}$, define

$$H(\tau) := F'(e^{\tau})e^{-4\tau} + F(e^{\tau})e^{-5\tau}$$

and define $\tilde{H}(\tau)$ analogously. Then $H, \tilde{H} \in L^\infty(\mathbb{R})$, and under the hypotheses of the main theorem, we have

$$H * w = \tilde{H} * w,$$

where

$$w(\tau) := [\text{some function in } L^1(\mathbb{R})].$$
Reduction to a convolution equation

The proof of this proposition involves considering a specific solution $u_{\text{lin}}$ of the linear wave equation with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$.

For $\alpha, \varepsilon > 0$, we define

$$u_{\text{lin}}^{\alpha,\varepsilon}(t, x) := \alpha u_{\text{lin}}((\alpha/\varepsilon)^2 t, (\alpha/\varepsilon)^2 x),$$

which solves the linear wave equation with initial data

$$(u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon}) := (u_{\text{lin}}^{\alpha,\varepsilon}(0), \partial_t u_{\text{lin}}^{\alpha,\varepsilon}(0)).$$

Under this rescaling,

$$\left\| (u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon}) \right\|_{\dot{H}^1 \times L^2} = \varepsilon \left\| (u_0, u_1) \right\|_{\dot{H}^1 \times L^2}.$$

In particular, if $F$ is an admissible nonlinearity for the NLW equation and $B_\eta$ is a corresponding ball given by the small-data scattering theorem, then $(u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon}) \in B_\eta$ for all $\varepsilon \ll \eta$. 
Reduction to a convolution equation

This solution will also have the property that

\[ u_{\text{lin}}(t, x) = \partial_t v_{\text{lin}}(t, x), \]

where \( v_{\text{lin}} \) is itself a solution of the linear wave equation with initial data \((v_0, v_1) \in \dot{H}^1 \times L^2\).

For \( \alpha, \varepsilon > 0 \), we define \( v_{\text{lin}}^{\alpha, \varepsilon} \) so that

\[ u_{\text{lin}}^{\alpha, \varepsilon}(t, x) = \partial_t v_{\text{lin}}^{\alpha, \varepsilon}(t, x). \]

Then \( v_{\text{lin}}^{\alpha, \varepsilon} \) solves the linear wave equation with initial data

\[ (v_0^{\alpha, \varepsilon}, v_1^{\alpha, \varepsilon}) := (v_{\text{lin}}^{\alpha, \varepsilon}(0), \partial_t v_{\text{lin}}^{\alpha, \varepsilon}(0)). \]

Under this rescaling,

\[ \|(v_0^{\alpha, \varepsilon}, v_1^{\alpha, \varepsilon})\|_{\dot{H}^1 \times L^2} = (\alpha/\varepsilon)^{-2}\varepsilon\|(v_0, v_1)\|_{\dot{H}^1 \times L^2}. \]
Reduction to a convolution equation

Proof (of the reduction)

Observe that

\[
\int_0^\infty \langle F(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t) \rangle_{L^2} \, dt
\]

\[
= \int_0^\infty \langle F(u_{\text{lin}}^{\alpha,\varepsilon}(t)), \partial_t v_{\text{lin}}^{\alpha,\varepsilon}(t) \rangle_{L^2} \, dt
\]

\[
= \int_0^\infty \left\langle \left[ \begin{array}{c} 0 \\ F(u_{\text{lin}}^{\alpha,\varepsilon}(t)) \end{array} \right], U(t) \left[ \begin{array}{c} v_0^{\alpha,\varepsilon} \\ v_1^{\alpha,\varepsilon} \end{array} \right] \right\rangle_{\dot{H}^1 \times L^2} \, dt
\]

\[
= \left\langle \int_0^\infty U(-t) \left[ \begin{array}{c} 0 \\ F(u_{\text{lin}}^{\alpha,\varepsilon}(t)) \end{array} \right] \, dt, \left[ \begin{array}{c} v_0^{\alpha,\varepsilon} \\ v_1^{\alpha,\varepsilon} \end{array} \right] \right\rangle_{\dot{H}^1 \times L^2}.
\]
Proof

Since $W_F((u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon})) = \hat{W}_\varepsilon((u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon}))$ (for all $\varepsilon \ll \eta, \tilde{\eta}$), we have

$$
\int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ F(u_{\text{lin}}^{\alpha,\varepsilon}(t)) \end{bmatrix} \, dt
= \int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ \tilde{F}(u_{\text{lin}}^{\alpha,\varepsilon}(t)) \end{bmatrix} \, dt + O \left( \left\| \begin{bmatrix} u_0^{\alpha,\varepsilon} \\ u_1^{\alpha,\varepsilon} \end{bmatrix} \right\|_{\dot{H}^1 \times L^2}^9 \right).
$$

Hence

$$
\int_0^\infty \langle F(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t) \rangle_{L^2} \, dt
= \int_0^\infty \langle \tilde{F}(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t) \rangle_{L^2} \, dt + O(\varepsilon^9) \left\| \begin{bmatrix} v_0^{\alpha,\varepsilon} \\ v_1^{\alpha,\varepsilon} \end{bmatrix} \right\|_{\dot{H}^1 \times L^2}
= \int_0^\infty \langle \tilde{F}(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t) \rangle_{L^2} \, dt + O_\alpha(\varepsilon^{12}).
$$
Reduction to a convolution equation

Proof

On the other hand, if $G(u) := F(u)u$, then

$$
\int_{0}^{\infty} \langle F(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t) \rangle_{L^2} dt
$$

$$
= \int_{0}^{\infty} \int_{\mathbb{R}^3} G(u_{\text{lin}}^{\alpha,\varepsilon}(t)) \, dx \, dt
$$

$$
= \int_{0}^{\infty} \int_{\mathbb{R}^3} \int_{0}^{\infty} u_{\text{lin}}^{\alpha,\varepsilon}(t) \, G'(\lambda) \, d\lambda \, dx \, dt
$$

$$
= \int_{0}^{\infty} G'(\lambda) \int_{0}^{\infty} \int_{\mathbb{R}^3} 1\{\lambda<u_{\text{lin}}^{\alpha,\varepsilon}(t,x)\} (t, x, \lambda) \, dx \, dt \, d\lambda
$$

$$
= \int_{0}^{\infty} G'(\lambda) \int_{0}^{\infty} \int_{\mathbb{R}^3} 1\{\lambda<u_{\text{lin}}(t,x)\} ((\alpha/\varepsilon)^2 t, (\alpha/\varepsilon)^2 x, \lambda/\alpha) \, dx \, dt \, d\lambda.
$$
Reduction to a convolution equation

Proof

Thus, if

\[ m(\lambda) := \left| \{(t, x) \in (0, \infty) \times \mathbb{R}^3 : u_{\text{lin}}(t, x) > \lambda \} \right| , \]

then

\[
\int_0^\infty \left< F(u_{\text{lin}}^{\alpha, \varepsilon}(t)), u_{\text{lin}}^{\alpha, \varepsilon}(t) \right>_{L^2} dt
\]

\[ = \int_0^\infty G'(\lambda) (\alpha/\varepsilon)^{-8} m(\lambda/\alpha) d\lambda \]

\[ = \frac{\varepsilon^8}{\alpha^8} \int_{-\infty}^\infty G'(e^\tau) e^\tau m(e^{\tau - \log \alpha}) d\tau \quad (\lambda =: e^\tau) \]

\[ = \cdots = \frac{16\pi \varepsilon^8}{3\alpha^2} (H \ast w)(\log 2\alpha), \]

where \( H(\tau) = G'(e^\tau)e^{-5\tau} \) and \( w(\tau) := \frac{12}{\pi} e^{-6\tau} m(e^{-(\tau - \log 2)}) \).
Reduction to a convolution equation

Proof

Finally, given a $\tau_0 \in \mathbb{R}$, let $\alpha := \frac{1}{2} e^{\tau_0}$ so that $\tau_0 = \log 2\alpha$. Combining the above, we deduce that

$$(H * w)(\tau_0) = (\tilde{H} * w)(\tau_0) + O(\varepsilon^4).$$

Taking $\varepsilon \to 0$, we arrive at the conclusion.
Reduction to a convolution equation

Task

Find a solution $u_{\text{lin}}$ of the linear wave equation with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$ for which:

- $u_{\text{lin}}(t, x) = \partial_t v_{\text{lin}}(t, x)$ (where $v_{\text{lin}}$ is a solution of the linear wave equation with initial data $(v_0, v_1) \in \dot{H}^1 \times L^2$)

- $w(\tau) = \frac{12}{\pi} e^{-6\tau} m(e^{-(\tau-\log 2)})$ is computable/analyzable (where $m(\lambda) = \left| \{ (t, x) \in (0, \infty) \times \mathbb{R}^3 : u_{\text{lin}}(t, x) > \lambda \} \right|$)

Approach

- Consider radially symmetric solutions, whose radial rescalings satisfy a 1D linear wave equation.

- Use d’Alembert’s formula to write the general solution of this equation and search for a suitable particular solution.
The radially symmetric solution

\[ u_{\text{lin}}(t, x) := \frac{f(r - t) - f(r + t)}{r}, \quad r := |x| \]

formed from the triangular function \( f(s) := \max\{1 - |s|, 0\} \) works.

After some computation, we find that

\[ w(\tau) = \left( e^{-3\tau} - \frac{4e^{-6\tau}}{(e^{-\tau} + 1)^3} \right) 1_{(0,\infty)}(\tau). \]
Deconvolutional determination of the nonlinearity
Now that we have $H \ast w = \tilde{H} \ast w$, we seek to formally “deconvolve” with $w$ to conclude that $H = \tilde{H}$, from which it will follow that $F = \tilde{F}$.

The tool that will enable us to do so is the following formulation of Wiener’s $L^1$ Tauberian theorem.

**Theorem (Wiener’s Tauberian theorem)**

Let $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. If $f \ast g = 0$ and $\hat{f}$ has no zeroes, then $g = 0$.

**Proposition**

Let $w$ be as defined previously. Then $\hat{w}$ has no zeroes.
Deconvolutional determination of the nonlinearity

Task

For the solution $u_{\text{lin}}$ found previously, ensure that $\hat{w}$ has no zeroes. In our case,

$$w(\tau) = \left( e^{-3\tau} - \frac{4e^{-6\tau}}{(e^{-\tau} + 1)^3} \right) 1_{(0,\infty)}(\tau).$$

Approach

- Decompose $w$ as $w = w_0 + w_1$ (since $\hat{w}$ does not seem to be explicitly computable).
- Compute $\hat{w}_0$, which has no zeroes, and show that $\hat{w}_1$ remains sufficiently small.
Proof (of main theorem)

We know that $H \ast w = \tilde{H} \ast w$. Wiener’s Tauberian theorem and the nonvanishing of $\hat{w}$ imply that $H = \tilde{H}$.

Retracing the definitions of $H$ and $\tilde{H}$, we conclude that $F = \tilde{F}$ (recall that $H(\tau) = G'(e^{\tau})e^{-5\tau}$, where $G(u) = F(u)u$).
Future work
Consider the Schrödinger equation in 1D with a (Schwartz) potential and a cubic-type nonlinearity:

\[
\begin{cases}
  i\partial_t u(t, x) = (-\Delta_x + V(x))u(t, x) + F(u(t, x)), & (t, x) \in \mathbb{R} \times \mathbb{R}; \\
  u(0, \cdot) = u_0.
\end{cases}
\]

Can the potential be determined from the scattering behaviour?
Future work: background

Let us first consider the linear Schrödinger equation

\[ i \partial_t u(t, x) = (-\Delta_x + V(x))u(t, x). \]

The stationary states of this equation are functions of space that solve the time-independent Schrödinger equation

\[ (-\Delta + V) f = k^2 f \quad \text{for some } k. \]

For \( k \in \mathbb{R} \setminus \{0\} \), let \( f_1(\cdot ; k) \) and \( f_2(\cdot ; k) \) denote the solutions of the latter that satisfy \( f_1(x; k) \sim e^{ikx} \) as \( x \to +\infty \) and \( f_2(x; k) \sim e^{-ikx} \) as \( x \to -\infty \) (called Jost solutions). Then there exist functions \( T \), \( R_1 \), and \( R_2 \) (called transmission and reflection coefficients) such that

\[ f_1(x; k) \sim \frac{1}{T(k)}e^{ikx} + \frac{R_2(k)}{T(k)}e^{-ikx} \quad \text{as } x \to -\infty, \]

\[ f_2(x; k) \sim \frac{1}{T(k)}e^{-ikx} + \frac{R_1(k)}{T(k)}e^{ikx} \quad \text{as } x \to +\infty. \]
In this setting, scattering behaviour is encoded by the scattering matrix

\[ S(k) := \begin{bmatrix} T(k) & R_2(k) \\ R_1(k) & T(k) \end{bmatrix}. \]

It is known that the scattering matrix is determined by \( R := R_1 \) (or \( R_2 \)) and the eigenvalues \(-\beta_n^2 < \cdots < -\beta_1^2 < 0\) of \( H \). These together with the constants \( \| f_1(x; i\beta_j) \|_{L_x}^{-2} \) defined by the corresponding eigenfunctions determine the potential (Faddeev, 1958).

However, \( S \) alone does not determine the potential (when \( H \) has eigenvalues) (Deift, 1978)!
Theorem (Deift–Trubowitz, 1979)

Let $\beta > \beta_n$ and define $g_\alpha := f_1(\cdot; i\beta) + \alpha f_2(\cdot; i\beta)$ for $\alpha > 0$. Then the reflection coefficient for the potential

$$V_\alpha := V - 2(\log g_\alpha)'$$

is $R_\alpha(k) := -\frac{k + i\beta}{k - i\beta} R(k)$ and the eigenvalues of

$$H_\alpha := -\Delta + V_\alpha$$

are $-\beta^2 < -\beta_n^2 < \cdots < -\beta_1^2 < 0$.

In particular, starting from the “vacuum potential” $V = 0$, one can construct a family of “reflectionless potentials”.
The nonlinearity might actually allow us to glean more information about the potential because, for instance, varying the amplitude of the initial data changes the solution nonlinearly.

In the focusing cubic case ($F(u) = -|u|^2 u$), we have access to soliton solutions. By scaling these so that they are sufficiently tall/narrow/fast, we might be able to “probe” the potential.
Thank you for your attention!