Convergence of the CP-AltLS algorithm for orthogonally and incoherently decomposable tensors

Nicholas Hu

The University of California, Los Angeles (UCLA)

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Introduction

Introduction

Tensors are multidimensional arrays that can be used to represent and analyze data, with the different dimensions/modes corresponding to different data components (e.g., position, time, frequency, intensity, object, type).

$$\mathcal{X} = \left[\begin{bmatrix} x_{111} & x_{121} \\ x_{211} & x_{221} \\ x_{311} & x_{321} \\ x_{411} & x_{421} \end{bmatrix} \quad \begin{bmatrix} x_{112} & x_{122} \\ x_{212} & x_{222} \\ x_{312} & x_{322} \\ x_{412} & x_{422} \end{bmatrix} \quad \begin{bmatrix} x_{113} & x_{123} \\ x_{213} & x_{223} \\ x_{313} & x_{323} \\ x_{413} & x_{423} \end{bmatrix} \right] \in \mathbb{R}^{4 \times 2 \times 3}$$

Vectors (first-order tensors) will be represented by *bold lowercase* letters (e.g., $\mathbf{a}$), matrices (second-order tensors) by *bold uppercase* letters (e.g., $\mathbf{A}$), and general tensors by *bold uppercase script* letters (e.g., $\mathcal{X}$).

The $(i_1, \ldots, i_N)$-entry of $\mathcal{X}$ will be denoted by $x_{i_1 \cdots i_N}$ and the j^{th} column of $\mathbf{A}$ by $\mathbf{a}_j$.

Introduction

Tensor decompositions are structured representations of tensors that render them more amenable to storage, manipulation, and/or analysis.

The **CP decomposition** expresses a tensor as a sum of simpler component tensors, which can reveal patterns in or features of the underlying data (e.g., for a tensor whose entries are excitation-emission intensities of chemical samples, each component tensor corresponds to a chemical compound in the samples).

$$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + \cdots + \mathcal{X}_R$$

Introduction: The CP decomposition

Recall that the eigendecomposition of a symmetric matrix $\mathbf{A} \in \mathbb{R}^{I \times I}$ is

$$\mathbf{A} = \sum_{i=1}^{I} \lambda_i \mathbf{u}_i \mathbf{u}_i^\top,$$

where $\lambda_i \in \mathbb{R}$ and $\mathbf{U} \in \mathbb{R}^{I \times I}$ is *orthogonal*.

Recall that the singular value decomposition of a matrix $\mathbf{A} \in \mathbb{R}^{I_1 \times I_2}$ is

$$\mathbf{A} = \sum_{i=1}^{\min\{I_1, I_2\}} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top,$$

where $\sigma_i \in \mathbb{R}_{\geq 0}$, and $\mathbf{U} \in \mathbb{R}^{I_1 \times \min\{I_1, I_2\}}$ and $\mathbf{V} \in \mathbb{R}^{I_2 \times \min\{I_1, I_2\}}$ have *orthonormal* columns.

Introduction: The CP decomposition

Definition (Outer product)

The **outer product** of $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ and $\mathcal{Y} \in \mathbb{R}^{J_1 \times \cdots \times J_M}$ is the tensor $\mathcal{X} \circ \mathcal{Y} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$ with entries

$$(\mathcal{X} \circ \mathcal{Y})_{i_1 \cdots i_N j_1 \cdots j_M} := x_{i_1 \cdots i_N} y_{j_1 \cdots j_M}.$$

Example $\mathbf{u} \in \mathbb{R}^I$, $\mathbf{v} \in \mathbb{R}^J$

$$\mathbf{u} \circ \mathbf{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_I \end{bmatrix} \circ \begin{bmatrix} v_1 \\ \vdots \\ v_J \end{bmatrix} = \begin{bmatrix} u_1 v_1 & \cdots & u_1 v_J \\ \vdots & \ddots & \vdots \\ u_I v_1 & \cdots & u_I v_J \end{bmatrix} (= \mathbf{u}\mathbf{v}^\top) \in \mathbb{R}^{I \times J}$$

Introduction: The CP decomposition

Definition (CP decomposition and rank)

A **CP decomposition** of $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ is a decomposition of the form

$$\mathcal{X} = [\![\boldsymbol{\lambda};\, \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}]\!] := \sum_{r=1}^{R} \lambda_r \mathbf{a}_r^{(1)} \circ \cdots \circ \mathbf{a}_r^{(N)},$$

where $\boldsymbol{\lambda} \in \mathbb{R}^R$ and $\mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R}$ has *unit-norm* (but not necessarily orthonormal) columns. The λ_r are called **weights** and the $\mathbf{A}^{(n)}$ are called **factor matrices**.

We say that $\mathcal{X}$ has **(CP) rank** R if R is *minimal* (note that every tensor has a CP decomposition with $R = \prod_n I_n$.)

Tensor rank agrees with matrix rank for second-order tensors (matrices) by the SVD/rank factorization.

However, tensor rank behaves very differently from matrix rank for higher-order tensors! For instance, we can have $\operatorname{rank}(\mathcal{X}) > \min_n I_n$.

Example

$$\mathcal{X} = \left[\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{\mathbf{X}^{(1)}} \; \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{\mathbf{X}^{(2)}} \right] \in \mathbb{R}^{2\times 2\times 2}$$

If $\mathcal{X} = \mathbf{a}_1 \circ \mathbf{b}_1 \circ \mathbf{c}_1 + \mathbf{a}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_2$, then $\mathbf{X}^{(i)} = \mathbf{A} \begin{bmatrix} c_{i1} & 0 \\ 0 & c_{i2} \end{bmatrix} \mathbf{B}^\top$.

In particular, since $\mathbf{X}^{(1)}$ is invertible, we would have $\mathbf{X}^{(2)}(\mathbf{X}^{(1)})^{-1} = \mathbf{A} \begin{bmatrix} c_{21}/c_{11} & 0 \\ 0 & c_{22}/c_{12} \end{bmatrix} \mathbf{A}^{-1}$. But $\mathbf{X}^{(2)}(\mathbf{X}^{(1)})^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is not diagonalizable, so $\operatorname{rank}(\mathcal{X}) > 2$.

Introduction: The CP decomposition

- The rank of a tensor over $\mathbb{R}$ may be greater than its rank over $\mathbb{C}$ even when its entries are real, as the previous example shows (Kruskal 1983).
- Determining whether the rank of a tensor is at most r is NP-complete in any finite field and NP-hard in any field $\mathbb{F} \supseteq \mathbb{Q}$, even for third-order tensors (Håstad 1990; Hillar and Lim 2013).
  [What is the rank (over $\mathbb{R}$) of the tensor in the previous example?]
  - In practice, the rank of a tensor is estimated by fitting CP decompositions (with varying R) to the tensor.
  - The best rank-r approximation of a tensor (in the Frobenius norm, for instance) may not exist (Paatero 2000).
- The maximum rank of a tensor with given dimensions is unknown in general.
  [What is the maximum rank (over $\mathbb{R}$) of a $2 \times 2 \times 2$ (real) tensor?]

Introduction: The CP decomposition

$$\mathcal{X} = [\![\boldsymbol{\lambda}\,;\, \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}]\!] \in \mathbb{R}^{I_1 \times \cdots \times I_N} \quad (\boldsymbol{\lambda} \in \mathbb{R}^R, \mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R})$$

Definition (Orthogonally decomposable tensor)

A tensor is **orthogonally decomposable** if it admits a CP decomposition whose factor matrices have *orthonormal* columns.

Such a decomposition must have $R \leq \min_n I_n$, so not all tensors are odeco!

Definition (Incoherently decomposable tensor)

The **coherence** of a matrix $\mathbf{A}$ with *unit-norm* columns is $\mu(\mathbf{A}) := \max_{i \neq j} |\langle \mathbf{a}_i, \mathbf{a}_j \rangle| \in [0, 1]$. A tensor is **$\mu$-coherently decomposable** if it admits a CP decomposition whose factor matrices have *coherence at most μ*.

Introduction: The CP-AltLS algorithm

$$\mathcal{X} = [\![\boldsymbol{\lambda}\,;\, \mathbf{A}^{(1)}, \dots, \mathbf{A}^{(N)}]\!] \in \mathbb{R}^{I_1 \times \cdots \times I_N} \quad (\boldsymbol{\lambda} \in \mathbb{R}^R, \mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R})$$

Given $\mathcal{X}$ and initial guesses for the $\mathbf{A}^{(n)}$, the **CP-alternating least squares** algorithm approximates each factor matrix successively by solving a least squares problem in which all other factor matrices are fixed:

$$\mathbf{A}^{(n)} = \operatorname*{argmin}_{\mathbf{A}} \left\| \mathcal{X} - [\![\boldsymbol{\lambda}\,;\, \mathbf{A}^{(1)}, \dots, \mathbf{A}^{(n-1)}, \mathbf{A}, \mathbf{A}^{(n+1)}, \dots, \mathbf{A}^{(N)}]\!] \right\|.$$

Definition (Inner product and norm of tensors)

The **(Frobenius) inner product** of tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ is $\langle \mathcal{X}, \mathcal{Y} \rangle := \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} x_{i_1 \cdots i_N} y_{i_1 \cdots i_N}$ and the **(Frobenius) norm** of $\mathcal{X}$ is $\|\mathcal{X}\| := \langle \mathcal{X}, \mathcal{X} \rangle^{\frac{1}{2}}$.

Introduction: The CP-AltLS algorithm

To solve these tensor least squares problems, we “reshape” the tensors into matrices.

Example

$$\mathcal{X} = \left[\begin{bmatrix} x_{111} & x_{121} \\ x_{211} & x_{221} \\ x_{311} & x_{321} \\ x_{411} & x_{421} \end{bmatrix} \begin{bmatrix} x_{112} & x_{122} \\ x_{212} & x_{222} \\ x_{312} & x_{322} \\ x_{412} & x_{422} \end{bmatrix} \begin{bmatrix} x_{113} & x_{123} \\ x_{213} & x_{223} \\ x_{313} & x_{323} \\ x_{413} & x_{423} \end{bmatrix}\right] \in \mathbb{R}^{4\times 2\times 3}$$

$$\mathbf{X}_{(1)} = \begin{bmatrix} \mathbf{x}_{:,1,1} & \mathbf{x}_{:,2,1} & \cdots & \mathbf{x}_{:,2,3} \end{bmatrix} = \begin{bmatrix} x_{111} & x_{121} & \cdots & x_{123} \\ x_{211} & x_{221} & \cdots & x_{223} \\ x_{311} & x_{321} & \cdots & x_{323} \\ x_{411} & x_{421} & \cdots & x_{423} \end{bmatrix} \in \mathbb{R}^{4\times 6}$$

$$\mathbf{X}_{(2)} = \begin{bmatrix} \mathbf{x}_{1,:,1} & \mathbf{x}_{2,:,1} & \cdots & \mathbf{x}_{4,:,3} \end{bmatrix} = \begin{bmatrix} x_{111} & x_{211} & \cdots & x_{413} \\ x_{121} & x_{221} & \cdots & x_{423} \end{bmatrix} \in \mathbb{R}^{2\times 12}$$

Introduction: The CP-AltLS algorithm

The matrix $\mathbf{X}_{(n)}$ is called the **mode-n matricization** of $\mathcal{X}$.

Example $\mathcal{X} = [\![\boldsymbol{\lambda}\,;\, \mathbf{A}, \mathbf{B}, \mathbf{C}]\!] \in \mathbb{R}^{4\times 2\times 3}$

$$x_{ijk} = \sum_{r=1}^{R} \lambda_r a_{ir} b_{jr} c_{kr}$$

$$\mathbf{x}_{:,j,k} = \sum_{r=1}^{R} \lambda_r \mathbf{a}_r b_{jr} c_{kr} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_R \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_R \end{bmatrix}}_{\boldsymbol{\Lambda}} \begin{bmatrix} b_{j1} c_{k1} \\ \vdots \\ b_{jR} c_{kR} \end{bmatrix}$$

$$\mathbf{X}_{(1)} = \mathbf{A}\boldsymbol{\Lambda} \begin{bmatrix} c_{11}\mathbf{b}_1^\top & c_{21}\mathbf{b}_1^\top & c_{31}\mathbf{b}_1^\top \\ \vdots & \vdots & \vdots \\ c_{1R}\mathbf{b}_R^\top & c_{2R}\mathbf{b}_R^\top & c_{3R}\mathbf{b}_R^\top \end{bmatrix} = \mathbf{A}\boldsymbol{\Lambda} \begin{bmatrix} (\mathbf{c}_1 \otimes \mathbf{b}_1)^\top \\ \vdots \\ (\mathbf{c}_R \otimes \mathbf{b}_R)^\top \end{bmatrix}.$$

Introduction: The CP-AltLS algorithm

Definition (Khatri–Rao product)

The **Khatri–Rao product** of $\mathbf{A} \in \mathbb{R}^{I \times K}$ and $\mathbf{B} \in \mathbb{R}^{J \times K}$ is the matrix $\mathbf{A} \odot \mathbf{B} \in \mathbb{R}^{IJ \times K}$ given by

$$\mathbf{A} \odot \mathbf{B} := \begin{bmatrix} \mathbf{a}_1 \otimes \mathbf{b}_1 & \cdots & \mathbf{a}_K \otimes \mathbf{b}_K \end{bmatrix} = \begin{bmatrix} a_{11}\mathbf{b}_1 & \cdots & a_{1K}\mathbf{b}_K \\ \vdots & \ddots & \vdots \\ a_{I1}\mathbf{b}_1 & \cdots & a_{IK}\mathbf{b}_K \end{bmatrix}.$$

<u>Example</u> $\mathcal{X} = [\![\boldsymbol{\lambda};\, \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}]\!] \in \mathbb{R}^{I_1 \times \cdots \times I_N}$

$$\boxed{\mathbf{X}_{(n)} = \mathbf{A}^{(n)} \boldsymbol{\Lambda} (\mathbf{A}^{(N)} \odot \cdots \odot \mathbf{A}^{(n+1)} \odot \mathbf{A}^{(n-1)} \odot \cdots \odot \mathbf{A}^{(1)})^\top}$$

Introduction: The CP-AltLS algorithm

Example $\mathbf{A} \in \mathbb{R}^{I \times K}$, $\mathbf{B} \in \mathbb{R}^{J \times K}$

$$(\mathbf{A} \odot \mathbf{B})^\top (\mathbf{A} \odot \mathbf{B}) = \begin{bmatrix} \mathbf{a}_1^\top \otimes \mathbf{b}_1^\top \\ \vdots \\ \mathbf{a}_K^\top \otimes \mathbf{b}_K^\top \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \otimes \mathbf{b}_1 & \cdots & \mathbf{a}_K \otimes \mathbf{b}_K \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{a}_1^\top \mathbf{a}_1 \otimes \mathbf{b}_1^\top \mathbf{b}_1 & \cdots & \mathbf{a}_1^\top \mathbf{a}_K \otimes \mathbf{b}_1^\top \mathbf{b}_K \\ \vdots & \ddots & \vdots \\ \mathbf{a}_K^\top \mathbf{a}_1 \otimes \mathbf{b}_K^\top \mathbf{b}_1 & \cdots & \mathbf{a}_K^\top \mathbf{a}_K \otimes \mathbf{b}_K^\top \mathbf{b}_K \end{bmatrix}$$

$$= (\mathbf{A}^\top \mathbf{A}) * (\mathbf{B}^\top \mathbf{B})$$

Definition (Hadamard product)

The **Hadamard product** of $\mathbf{A} \in \mathbb{R}^{I \times J}$ and $\mathbf{B} \in \mathbb{R}^{I \times J}$ is the matrix $\mathbf{A} * \mathbf{B} \in \mathbb{R}^{I \times J}$ with entries $(\mathbf{A} * \mathbf{B})_{ij} := a_{ij} b_{ij}$.

Introduction: The CP-AltLS algorithm

Returning to the least squares problem,

$$\begin{aligned}
\mathbf{A}^{(n)} &= \operatorname*{argmin}_{\mathbf{A}} \left\| \mathcal{X} - [\![\boldsymbol{\lambda}\,;\, \mathbf{A}^{(1)}, \dots, \mathbf{A}^{(n-1)}, \mathbf{A}, \mathbf{A}^{(n+1)}, \dots, \mathbf{A}^{(N)}]\!] \right\| \\
&= \operatorname*{argmin}_{\mathbf{A}} \left\| \mathbf{X}_{(n)} - \mathbf{A}\boldsymbol{\Lambda}(\mathbf{K}^{(n)})^\top \right\|,
\end{aligned}$$

where $\mathbf{K}^{(n)} := \bigodot_{m \neq n} \mathbf{A}^{(m)}$. Hence

$$\begin{aligned}
\mathbf{A}^{(n)}\boldsymbol{\Lambda} &= \mathbf{X}_{(n)}((\mathbf{K}^{(n)})^\top)^+ \\
&= \mathbf{X}_{(n)}\mathbf{K}^{(n)}((\mathbf{K}^{(n)})^\top \mathbf{K}^{(n)})^+ \\
&= \mathbf{X}_{(n)}\mathbf{K}^{(n)}(\mathbf{H}^{(n)})^+,
\end{aligned}$$

where $\mathbf{H}^{(n)} := \ast_{m \neq n} \mathbf{G}^{(m)}$ and $\mathbf{G}^{(m)} := (\mathbf{A}^{(m)})^\top(\mathbf{A}^{(m)})$.

After solving for $\mathbf{A}^{(n)}\boldsymbol{\Lambda}$, we normalize its columns to obtain $\mathbf{A}^{(n)}$ and update $\boldsymbol{\lambda}$ accordingly.

Algorithm: CP-AltLS

Input: $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, $\mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R}$ $(n \in [N])$

for $n = 1$ **to** N

  $\mathbf{G}^{(n)} \leftarrow (\mathbf{A}^{(n)})^\top \mathbf{A}^{(n)}$

while *stopping condition has not been satisfied*

  for $n = 1$ **to** N

    $\mathbf{K}^{(n)} \leftarrow \bigodot_{m \neq n} \mathbf{A}^{(m)}$

    $\mathbf{H}^{(n)} \leftarrow \ast_{m \neq n} \mathbf{G}^{(m)}$

    $\mathbf{A}^{(n)} \leftarrow \mathbf{X}_{(n)} \mathbf{K}^{(n)} (\mathbf{H}^{(n)})^+$

    normalize columns of $\mathbf{A}^{(n)}$, updating $\boldsymbol{\lambda}$ accordingly

    $\mathbf{G}^{(n)} \leftarrow (\mathbf{A}^{(n)})^\top \mathbf{A}^{(n)}$

Output: $\boldsymbol{\lambda} \in \mathbb{R}^R$, $\mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R}$ $(n \in [N])$

Introduction: The CP-AltLS algorithm

- Existing results on CP-AltLS convergence are *limited in applicability* (e.g., to rank-one (Wang and Chu 2014) or orthogonal (Wang, Chu, and Yu 2015) decompositions), *qualitative*, *non-explicit*, and/or otherwise imprecise.
- Uschmajew proved local linear convergence to local minima of the (reconstruction) error satisfying a nondegeneracy condition, and established a quantitative sufficient condition for nondegeneracy in the rank-one case (Uschmajew 2012).
- Other methods for computing CP decompositions (e.g., the higher-order power method) have been studied, also under rank-one (Hu and Li 2018) or orthogonality assumptions (Hu and Ye 2023).

Introduction: The CP-AltLS algorithm

Our result (· , Iwen, Needell, and Wang 2025):

- arbitrary rank
- orthogonal and incoherent decompositions
- quantitative and explicit
- specific order of convergence
- more direct and less technical proof

Convergence of CP-AltLS

Convergence of CP-AltLS

We consider the "angular error" in the approximation of the exact factor matrices $\mathbf{A}^{(n)}$ by the approximate factor matrices $\mathbf{A}^{(n,k)}$ after k iterations:

$$\varepsilon_k := \max_{n\in[N],\, r\in[R]} \left|\sin \angle(\mathbf{a}_r^{(n,k)}, \mathbf{a}_r^{(n)})\right|.$$

Iteration numbers will be appended to the superscripts of vectors and matrices produced by the algorithm (e.g., $\boldsymbol{\lambda}^{(k)}$ will denote $\boldsymbol{\lambda}$ after k iterations, $\mathbf{A}^{(n,k)}$ will denote $\mathbf{A}^{(n)}$ after k iterations).

The weights $\boldsymbol{\lambda}^{(k)}$ will also converge *up to signs*, though we will not discuss this here.

Convergence of CP-AltLS

We will begin by analyzing a simplified version of CP-AltLS.

Algorithm: "Decoupled" CP-AltLS

Input: $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, $\mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R}$ $(n \in [N])$

while *stopping condition has not been satisfied*

- **for** $n = 1$ **to** N
  - $\mathbf{G}^{(n)} \leftarrow (\mathbf{A}^{(n)})^\top \mathbf{A}^{(n)}$
  - $\mathbf{K}^{(n)} \leftarrow \bigodot_{m \neq n} \mathbf{A}^{(m)}$
- **for** $n = 1$ **to** N
  - $\mathbf{H}^{(n)} \leftarrow \ast_{m \neq n} \mathbf{G}^{(m)}$
  - $\mathbf{A}^{(n)} \leftarrow \mathbf{X}_{(n)} \mathbf{K}^{(n)} (\mathbf{H}^{(n)})^+$
  - normalize columns of $\mathbf{A}^{(n)}$, updating $\boldsymbol{\lambda}$ accordingly

Output: $\boldsymbol{\lambda} \in \mathbb{R}^R$, $\mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R}$ $(n \in [N])$

> $\mathbf{A}^{(n,k)}$ depends only on $\mathbf{A}^{(1,k-1)}, \ldots, \mathbf{A}^{(n-1,k-1)}, \mathbf{A}^{(n+1,k-1)}, \ldots, \mathbf{A}^{(N,k-1)}$
> instead of
> $\mathbf{A}^{(1,k)}, \ldots, \mathbf{A}^{(n-1,k)}, \mathbf{A}^{(n+1,k-1)}, \ldots, \mathbf{A}^{(N,k-1)}$

Convergence of CP-AltLS: Odeco tensors

Theorem (Local convergence of CP-AltLS for odeco tensors)

Suppose that $\mathcal{X} = [\![\boldsymbol{\lambda}\,;\, \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}]\!]$ ($N \geq 3$) *for some* $\boldsymbol{\lambda} \in \mathbb{R}^R$ *and some* $\mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R}$ *with* <u>*orthonormal columns*</u>*. In addition, let* $\kappa := \frac{\max_{r \in [R]} |\lambda_r|}{\min_{r \in [R]} |\lambda_r|}$.

Then there exists an absolute constant $C > 0$ *such that if* $\varepsilon_0 < C\kappa^{-\frac{1}{N-2}} R^{-\frac{1}{N-1}}$, *then* $\varepsilon_k \to 0$ *with* <u>*order* $N-1$</u> (*more precisely,* $\varepsilon_k \leq \alpha^{(N-1)^k}$ *for some absolute constant* $0 < \alpha < 1$).

Convergence of CP-AltLS: Odeco tensors

Proof (outline)

Consider

$$\mathbf{A}^{(n,k)} = \mathbf{X}_{(n)}\mathbf{K}^{(n,k)}(\mathbf{H}^{(n,k)})^{+} = \mathbf{A}^{(n)}\mathbf{\Lambda}(\mathbf{K}^{(n)})^{\top}\mathbf{K}^{(n,k)}(\mathbf{H}^{(n,k)})^{+}.$$

WLOG assume that $\langle \mathbf{a}_r^{(n,k)}, \mathbf{a}_r^{(n)} \rangle \geq 0$ for all k, n, r by adjusting $\boldsymbol{\lambda}$.

- $\varepsilon_{k-1} \ll 1 \implies \mathbf{A}^{(m,k-1)} \approx \mathbf{A}^{(m)}\ (\forall m)$
- $\mathbf{G}^{(m,k)} = (\mathbf{A}^{(m,k-1)})^{\top}(\mathbf{A}^{(m,k-1)}) \approx (\mathbf{A}^{(m)})^{\top}\mathbf{A}^{(m)} = \mathbf{I}$
- $(\mathbf{K}^{(n)})^{\top}\mathbf{K}^{(n,k)} = \circledast_{m\neq n}(\mathbf{A}^{(m)})^{\top}\mathbf{A}^{(m,k-1)} \approx \circledast_{m\neq n}\mathbf{I} = \mathbf{I}$
- $\mathbf{H}^{(n,k)} = \circledast_{m\neq n}\mathbf{G}^{(m,k)} \approx \circledast_{m\neq n}\mathbf{I} = \mathbf{I}$
- $\mathbf{A}^{(n,k)} = \mathbf{A}^{(n)}\mathbf{\Lambda}(\mathbf{K}^{(n)})^{\top}\mathbf{K}^{(n,k)}(\mathbf{H}^{(n,k)})^{+} \approx \mathbf{A}^{(n)}\mathbf{\Lambda}\,\mathbf{I}\,\mathbf{I} = \mathbf{A}^{(n)}\mathbf{\Lambda}$
- $\mathbf{A}^{(n,k)} \approx \mathbf{A}^{(n)}\mathbf{\Lambda}\ (\forall n) \implies \varepsilon_k \ll 1$

Convergence of CP-AltLS: Odeco tensors

Proof (sketch)

Let us write $\mathbf{A} = [\mathbf{D}, \mathbf{D}']$ to indicate that $\mathbf{D}$ and $\mathbf{D}'$ are the diagonal and off-diagonal parts of $\mathbf{A}$.

If $\varepsilon_{k-1} = \max_{n,r} |\sin \angle(\mathbf{a}_r^{(n,k-1)}, \mathbf{a}_r^{(n)})| \ll 1$, then

$$\mathbf{a}_r^{(m,k-1)} = \mathbf{a}_r^{(m)} + \mathcal{O}(\varepsilon_{k-1}) \quad \text{for all } m, r,$$

so

$$(\mathbf{A}^{(m)})^\top \mathbf{A}^{(m,k-1)} = [\mathbf{I} + \mathcal{O}(\varepsilon_{k-1}), \mathcal{O}(\varepsilon_{k-1})] \quad \text{for all } m.$$

Similarly,

$$\mathbf{G}^{(m,k)} = (\mathbf{A}^{(m,k-1)})^\top (\mathbf{A}^{(m,k-1)}) = [\mathbf{I}, \mathcal{O}(\varepsilon_{k-1})] \quad \text{for all } m.$$

Convergence of CP-AltLS: Odeco tensors

Proof (sketch)

Next,

$$
\begin{aligned}
(\mathbf{K}^{(n)})^\top \mathbf{K}^{(n,k)} &= \mathop{\Large\ast}_{m \neq n} (\mathbf{A}^{(m)})^\top \mathbf{A}^{(m,k-1)} \\
&= \mathop{\Large\ast}_{m \neq n} [\mathbf{I} + \mathcal{O}(\varepsilon_{k-1}), \mathcal{O}(\varepsilon_{k-1})] \\
&= [\mathbf{I} + \mathcal{O}(\varepsilon_{k-1}), \mathcal{O}(\varepsilon_{k-1}^{N-1})]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{H}^{(n,k)} &= \mathop{\Large\ast}_{m \neq n} \mathbf{G}^{(m,k)} \\
&= \mathop{\Large\ast}_{m \neq n} [\mathbf{I}, \mathcal{O}(\varepsilon_{k-1})] \\
&= [\mathbf{I}, \mathcal{O}(\varepsilon_{k-1}^{N-1})].
\end{aligned}
$$

Convergence of CP-AltLS: Odeco tensors

Proof (sketch)

By a geometric series argument,

$$(\mathbf{H}^{(n,k)})^{+} = (\mathbf{I} + \mathcal{O}(\varepsilon_{k-1}^{N-1}))^{-1} = [\mathbf{I} + \mathcal{O}(\varepsilon_{k-1}^{N-1}), \mathcal{O}(\varepsilon_{k-1}^{N-1})].$$

Finally,

$$\begin{aligned}\mathbf{A}^{(n,k)} &= \mathbf{A}^{(n)}\mathbf{\Lambda}\,(\mathbf{K}^{(n)})^{\top}\mathbf{K}^{(n,k)}(\mathbf{H}^{(n,k)})^{+} \\ &= \mathbf{A}^{(n)}\mathbf{\Lambda}\,[\mathbf{I} + \mathcal{O}(\varepsilon_{k-1}), \mathcal{O}(\varepsilon_{k-1}^{N-1})]\,[\mathbf{I} + \mathcal{O}(\varepsilon_{k-1}^{N-1}), \mathcal{O}(\varepsilon_{k-1}^{N-1})] \\ &= \mathbf{A}^{(n)}\mathbf{\Lambda}\,[\mathbf{I} + \mathcal{O}(\varepsilon_{k-1}), \mathcal{O}(\varepsilon_{k-1}^{N-1})],\end{aligned}$$

which implies that $\varepsilon_k = \mathcal{O}(\varepsilon_{k-1}^{N-1})$.

Convergence of CP-AltLS: Odeco tensors

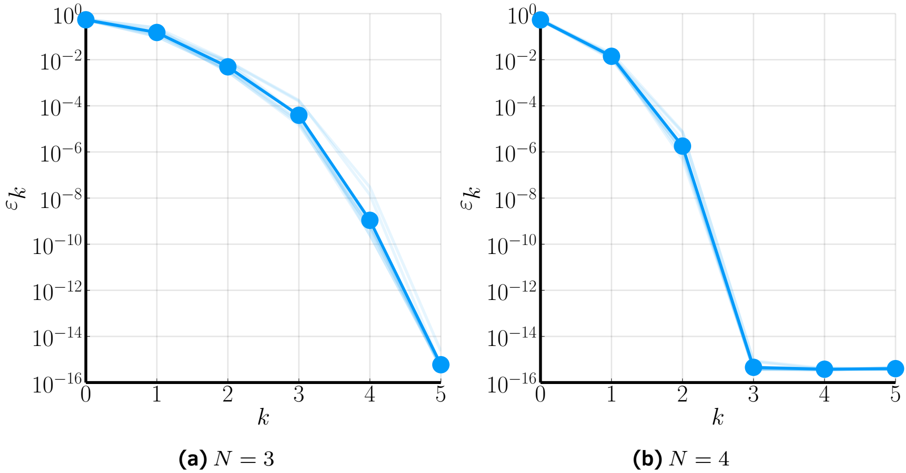


Figure: *Convergence of (decoupled) CP-AltLS for N^{th}-order odeco tensors.*

Convergence of CP-AltLS: Ideco tensors

Theorem (Local convergence of CP-AltLS for ideco tensors)

Suppose that $\mathcal{X} = [\![\boldsymbol{\lambda}\,;\, \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}]\!]$ ($N \geq 3$) *for some* $\boldsymbol{\lambda} \in \mathbb{R}^R$ *and some* $\mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R}$ *with* <u>*normalized columns*</u> *and* <u>*coherence at most* μ</u>*. In addition, let* $\kappa := \frac{\max_{r \in [R]} |\lambda_r|}{\min_{r \in [R]} |\lambda_r|}$*.*

Then there exists an absolute constant $C > 0$ *such that if* $\max\{\varepsilon_0, \mu\} < C\kappa^{-\frac{1}{N-2}} R^{-\frac{2}{N-2}}$*, then* $\varepsilon_k \to 0$ <u>*linearly*</u> *(more precisely,* $\varepsilon_k \leq \rho \varepsilon_{k-1}$ *for some absolute constant* $0 < \rho < 1$*).*

Convergence of CP-AltLS: Ideco tensors

Proof (outline)

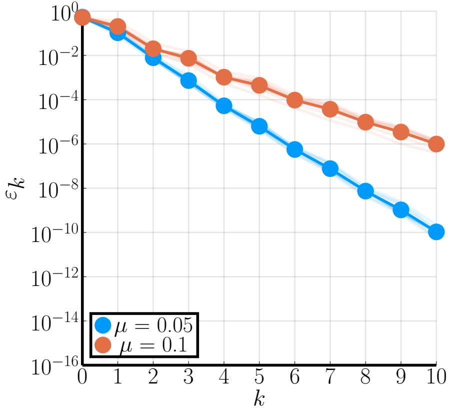
WLOG assume that $\langle \mathbf{a}_r^{(n,k)}, \mathbf{a}_r^{(n)} \rangle \geq 0$ for all k, n, r by adjusting $\boldsymbol{\lambda}$.

We no longer have $(\mathbf{A}^{(m)})^\top \mathbf{A}^{(m)} = \mathbf{I}$, so let us define

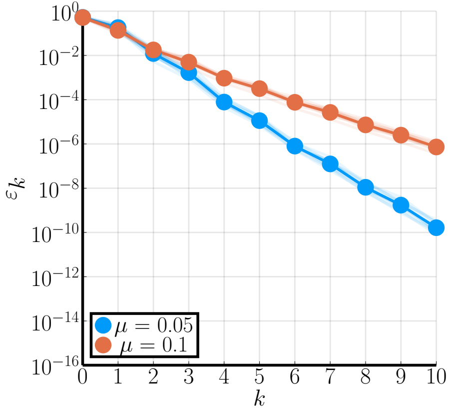
$$\mathbf{G}^{(m)} := (\mathbf{A}^{(m)})^\top \mathbf{A}^{(m)} \quad \text{and} \quad \mathbf{H}^{(n)} := \mathop{\ast}_{m \neq n} \mathbf{G}^{(m)}.$$

- $\varepsilon_{k-1} \ll 1 \implies \mathbf{A}^{(m,k-1)} \approx \mathbf{A}^{(m)} \ (\forall m)$
- $\mathbf{G}^{(m,k)} = (\mathbf{A}^{(m,k-1)})^\top (\mathbf{A}^{(m,k-1)}) \approx (\mathbf{A}^{(m)})^\top \mathbf{A}^{(m)} = \mathbf{G}^{(m)}$
- $(\mathbf{K}^{(n)})^\top \mathbf{K}^{(n,k)} = \ast_{m \neq n} (\mathbf{A}^{(m)})^\top \mathbf{A}^{(m,k-1)} \approx \ast_{m \neq n} \mathbf{G}^{(m)} = \mathbf{H}^{(n)}$
- $\mathbf{H}^{(n,k)} = \ast_{m \neq n} \mathbf{G}^{(m,k)} \approx \ast_{m \neq n} \mathbf{G}^{(m)} = \mathbf{H}^{(n)}$
- $\mathbf{A}^{(n,k)} = \mathbf{A}^{(n)} \boldsymbol{\Lambda} (\mathbf{K}^{(n)})^\top \mathbf{K}^{(n,k)} (\mathbf{H}^{(n,k)})^+ \approx \mathbf{A}^{(n)} \boldsymbol{\Lambda} \mathbf{H}^{(n)} (\mathbf{H}^{(n)})^+ = \mathbf{A}^{(n)} \boldsymbol{\Lambda}$
- $\mathbf{A}^{(n,k)} \approx \mathbf{A}^{(n)} \boldsymbol{\Lambda} \ (\forall n) \implies \varepsilon_k \ll 1$

Convergence of CP-AltLS: Ideco tensors



(a) $N = 3$



(b) $N = 4$

Figure: *Convergence of (decoupled) CP-AltLS for N^{th}-order ideco tensors.*

Algorithm: CP-AltLS

Input: $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, $\mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R}$ $(n \in [N])$

for $n = 1$ **to** N

  $\mathbf{G}^{(n)} \leftarrow (\mathbf{A}^{(n)})^\top \mathbf{A}^{(n)}$

while *stopping condition has not been satisfied*

  for $n = 1$ **to** N

    $\mathbf{K}^{(n)} \leftarrow \bigodot_{m \neq n} \mathbf{A}^{(m)}$

    $\mathbf{H}^{(n)} \leftarrow \mathop{\ast}_{m \neq n} \mathbf{G}^{(m)}$

    $\mathbf{A}^{(n)} \leftarrow \mathbf{X}_{(n)} \mathbf{K}^{(n)} (\mathbf{H}^{(n)})^+$

    normalize columns of $\mathbf{A}^{(n)}$, updating $\boldsymbol{\lambda}$ accordingly

    $\mathbf{G}^{(n)} \leftarrow (\mathbf{A}^{(n)})^\top \mathbf{A}^{(n)}$

Output: $\boldsymbol{\lambda} \in \mathbb{R}^R$, $\mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R}$ $(n \in [N])$

In the standard CP-AltLS algorithm, we have

$$\mathbf{K}^{(n,k)} = \left(\bigodot_{m>n} \mathbf{A}^{(m,k-1)} \right) \odot \left(\bigodot_{m<n} \mathbf{A}^{(m,k)} \right) \text{ instead of } \bigodot_{m \neq n} \mathbf{A}^{(m,k-1)}.$$

Similarly,

$$\mathbf{H}^{(n,k)} = \left(\mathop{\Large\ast}_{m>n} \mathbf{G}^{(m,k-1)} \right) \ast \left(\mathop{\Large\ast}_{m<n} \mathbf{G}^{(m,k)} \right) \text{ instead of } \mathop{\Large\ast}_{m \neq n} \mathbf{G}^{(m,k-1)},$$

where

$$\mathbf{G}^{(m,k)} = (\mathbf{A}^{(m,k)})^\top (\mathbf{A}^{(m,k)}) \text{ instead of } (\mathbf{A}^{(m,k-1)})^\top (\mathbf{A}^{(m,k-1)}).$$

Convergence of CP-AltLS: The standard algorithm

To account for this, we consider the angular error of each individual mode $m \in [N]$,

$$\varepsilon_{m,k} := \max_{r \in [R]} |\sin \angle(\mathbf{a}_r^{(m,k)}, \mathbf{a}_r^{(m)})|,$$

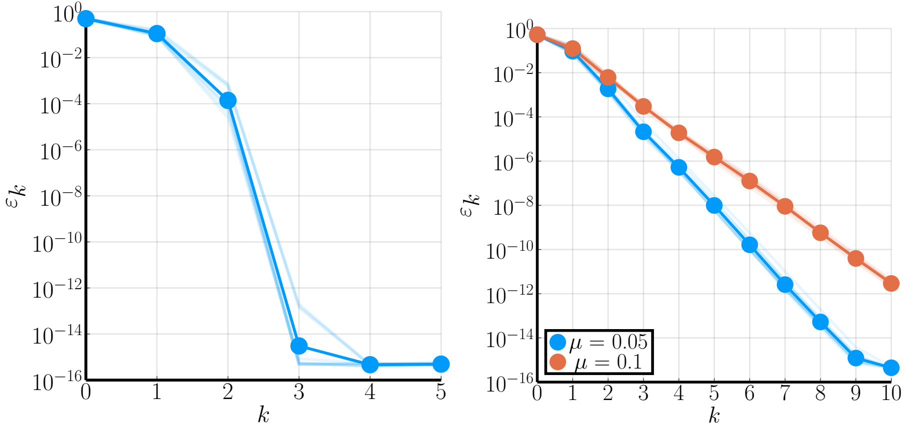
and *assume inductively* that

$$\varepsilon_{m,k} \leq \varepsilon_{k-1} = \max_{m \in [N]} \varepsilon_{m,k-1} \quad \text{for all } m < n.$$

That is, in the computation of $\mathbf{A}^{(n,k)}$, we assume that the factor matrices $\mathbf{A}^{(m,k)}$ computed in the *current* iteration for $m < n$ are *no less accurate than* the factor matrices $\mathbf{A}^{(m,k-1)}$ computed in the *previous* iteration.

Thus, the results for the decoupled algorithm also hold for the standard algorithm.

Convergence of CP-AltLS: The standard algorithm



(a) *Odeco tensor* **(b)** *Ideco tensors*

Figure: *Convergence of CP-AltLS for 3^{rd}-order tensors.*

In fact, the standard algorithm applied to odeco tensors appears to converge with an order *greater than* $N - 1$... why?

Examining the convergence proof, we find that instead of $\varepsilon_k = \mathcal{O}(\varepsilon_{k-1}^{N-1})$, we have

$$\varepsilon_{n,k} = \mathcal{O}(\varepsilon_{N,k-1} \cdots \varepsilon_{n+1,k-1} \cdot \varepsilon_{n-1,k} \cdots \varepsilon_{1,k}).$$

For example, if $N = 3$ and $\varepsilon_0 = \max\{\varepsilon_{1,0}, \varepsilon_{2,0}, \varepsilon_{3,0}\} = \mathcal{O}(\varepsilon)$, then

$$\begin{aligned}
\varepsilon_{1,1} &= \mathcal{O}(\varepsilon \cdot \varepsilon) &&= \mathcal{O}(\varepsilon^2) \\
\varepsilon_{2,1} &= \mathcal{O}(\varepsilon \cdot \varepsilon^2) &&= \mathcal{O}(\varepsilon^3) \\
\varepsilon_{3,1} &= \mathcal{O}(\varepsilon^3 \cdot \varepsilon^2) &&= \mathcal{O}(\varepsilon^5) \\
\varepsilon_{1,2} &= \mathcal{O}(\varepsilon^5 \cdot \varepsilon^3) &&= \mathcal{O}(\varepsilon^8) \\
&\vdots && \vdots
\end{aligned}$$

Convergence of CP-AltLS: The standard algorithm

Hence

$$\begin{aligned} \varepsilon_1 &= \max\{\mathcal{O}(\varepsilon^2), \mathcal{O}(\varepsilon^3), \mathcal{O}(\varepsilon^5)\} &&= \mathcal{O}(\varepsilon^2) \\ \varepsilon_2 &= \max\{\mathcal{O}(\varepsilon^8), \mathcal{O}(\varepsilon^{13}), \mathcal{O}(\varepsilon^{21})\} &&= \mathcal{O}(\varepsilon^8) \\ \vdots & \qquad\qquad\qquad \vdots && \quad \vdots \end{aligned}$$

The order of convergence is $q^3 \approx 4.2$, where $q = \frac{1+\sqrt{5}}{2} \approx 1.6$ is the positive solution of $1 + q = q^2$.

Theorem (Local convergence of CP-AltLS for odeco tensors)

Suppose that $\mathcal{X} = [\![\boldsymbol{\lambda}; \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}]\!]$ ($N \geq 3$) …

… *if* $\varepsilon_0 < C\kappa^{-\frac{1}{N-2}} R^{-\frac{1}{N-1}}$*, then* $\varepsilon_k \to 0$ *with order* q^N*, where* q *is the (unique) positive solution of* $1 + \cdots + q^{N-3} + q^{N-2} = q^{N-1}$.

Convergence of CP-AltLS: The standard algorithm

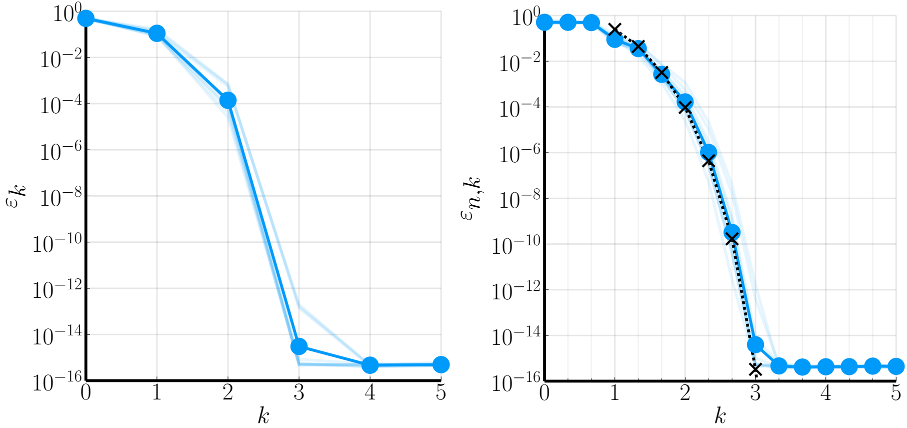


Figure: *Convergence of CP-AltLS for a 3^{rd}-order odeco tensor. The $\times$'s are $\varepsilon_{3,k-1} \cdots \varepsilon_{n+1,k-1} \cdot \varepsilon_{n-1,k} \cdots \varepsilon_{1,k}$ (that is, the products of the angular errors in the two preceding subiterations).*

Conclusion

Conclusion – summary

Our result (· , Iwen, Needell, and Wang 2025):

- arbitrary rank: not limited to rank-one tensors
- orthogonal and incoherent decompositions: not limited to odeco tensors
- quantitative and explicit: inequalities with computable constants
- specific order of convergence: order q^N (where $1 + \cdots + q^{N-2} = q^{N-1}$) for N^{th}-order odeco tensors, order 1 for ideco tensors
- more direct and less technical proof: matrix analysis

Conclusion – future work

- Convergence acceleration through factor matrix orthogonalization (Sharan and Valiant 2017) or coherence reduction
- Tensor recovery from "noisy measurements" (e.g., recovering $\mathcal{X}$ from $\mathcal{A}(\mathcal{X}) + \mathcal{E}$ for some linear operator $\mathcal{A}$ and some unknown/random tensor $\mathcal{E}$)
- Convergence of AltLS for other tensor decompositions (e.g., Tucker, DEDICOM)

Thank you for your attention!