Projection methods for linear systems

Nicholas Hu · Last updated on 2025-03-14

Let $A \in \mathbb{R}^{n \times n}$ be invertible and $b \in \mathbb{R}^n$. A **projection method** for solving $Ax = b$ produces an approximation $\tilde{x}$ to the exact solution x^* within an m-dimensional **search subspace** $\mathcal{K}$ translated by an initial guess $x^{(0)}$, such that the **residual** $b - A\tilde{x}$ is *orthogonal to* an m-dimensional **constraint subspace** $\mathcal{L}$. In other words, $\tilde{x} \in x^{(0)} + \mathcal{K}$ with $b - A\tilde{x} \in \mathcal{L}^\perp$. If $\mathcal{L} = \mathcal{K}$, the projection method is said to be **orthogonal** and its orthogonality constraints are known as the **Galerkin conditions**; otherwise, the method is said to be **oblique** and its constraints are known as the **Petrov–Galerkin conditions**.

Such a method is well-defined if and only if $A\mathcal{K} \cap \mathcal{L}^\perp = \{0\}$. Indeed, if $A\mathcal{K} \cap \mathcal{L}^\perp = \{0\}$ and $V, W \in \mathbb{R}^{n \times m}$ are matrices whose columns are bases of $\mathcal{K}$ and $\mathcal{L}$, then we must have $\tilde{x} = x^{(0)} + Vy$ for some $y \in \mathbb{R}^m$ such that $W^\top(r^{(0)} - AVy) = 0$, where $r^{(0)} := b - Ax^{(0)}$. Hence

$$\tilde{x} = x^{(0)} + V(W^\top AV)^{-1}W^\top r^{(0)},$$

where $W^\top AV$ is invertible because $\operatorname{im}(AV) \cap \ker(W^\top) = \{0\}$. In addition, if $\tilde{x}' \in x^{(0)} + \mathcal{K}$ with $b - A\tilde{x}' \in \mathcal{L}^\perp$, then $A(\tilde{x} - \tilde{x}') \in A\mathcal{K} \cap \mathcal{L}^\perp$, so $\tilde{x} = \tilde{x}'$. Conversely, if the method is well-defined and $Av \in \mathcal{L}^\perp$ for some $v \in \mathcal{K}$, then $\tilde{x} + v \in x^{(0)} + \mathcal{K}$ and $b - A(\tilde{x} + v) \in \mathcal{L}^\perp$, so $v = 0$ and hence $Av = 0$.

This projection process may be iterated by selecting new subspaces $\mathcal{K}$ and $\mathcal{L}$ and using $\tilde{x}$ as the initial guess for the next iteration, yielding a variety of iterative methods for linear systems, such as the well-known Krylov subspace methods. These iterative methods can sometimes experience a "lucky breakdown" when the projection produces the exact solution:

> If $r^{(0)} \in \mathcal{K}$ and $\mathcal{K}$ is A-invariant, then $A\tilde{x} = b$ (or equivalently, $\tilde{x} = x^*$).

Proof. By definition, $\tilde{x} - x^{(0)} \in \mathcal{K}$ and $b - A\tilde{x} \in \mathcal{L}^\perp$. On the other hand, $A\mathcal{K} \subseteq \mathcal{K}$ and $\dim(A\mathcal{K}) = \dim(\mathcal{K})$ since A is invertible, so $A\mathcal{K} = \mathcal{K}$. Hence $b - A\tilde{x} = r^{(0)} - A(\tilde{x} - x^{(0)}) \in A\mathcal{K} \cap \mathcal{L}^\perp = \{0\}$. ∎

Error projection methods

An **error projection method** is a projection method where A is symmetric positive definite (SPD) and $\mathcal{L} = \mathcal{K}$. Such methods are well-defined because if $Av \in \mathcal{K}^\perp$ for some $v \in \mathcal{K}$, then $\|v\|_A^2 = 0$.

> If A is SPD and $\mathcal{L} = \mathcal{K}$, then $\tilde{x}$ uniquely minimizes the A-norm of the **error** $x^* - \tilde{x}$ over $x^{(0)} + \mathcal{K}$.

Proof. For all $x \in x^{(0)} + \mathcal{K}$, we have $\|x^* - x\|_A^2 = \|x^* - \tilde{x}\|_A^2 + \|\tilde{x} - x\|_A^2$ because $\tilde{x} - x \in \mathcal{K}$ and $x^* - \tilde{x} \perp_A \mathcal{K}$ according to the Galerkin conditions. ∎

The gradient descent method

The **gradient descent method** for solving $Ax = b$ when A is SPD is the iterative method with $\mathcal{K} = \mathcal{L} := \operatorname{span}\{r^{(k)}\}$, where $x^{(k)}$ denotes the k^{th} iterate and $r^{(k)} := b - Ax^{(k)}$. Thus, $x^{(k+1)}$ minimizes the A-norm of the error over the line $x^{(k)} + \operatorname{span}\{r^{(k)}\}$; indeed, if $f(x) := \frac{1}{2}\|x^* - x\|_A^2$, then $\nabla f(x^{(k)}) = -r^{(k)}$, so $r^{(k)}$ represents the direction of steepest descent of f. The projection formula above reduces to

$$x^{(k+1)} = x^{(k)} + \frac{\langle r^{(k)}, r^{(k)} \rangle}{\langle Ar^{(k)}, r^{(k)} \rangle} r^{(k)} =: x^{(k)} + \alpha_k r^{(k)}.$$

We also note that $r^{(k+1)} = r^{(k)} - \alpha_k A r^{(k)}$, so this method can be implemented with only one multiplication by A per iteration.

To analyze the convergence of the gradient descent method, we consider the error $e^{(k)} := x^* - x^{(k)}$. Using the fact that $e^{(k+1)} = e^{(k)} - \alpha_k r^{(k)} \perp_A r^{(k)}$, we compute that

$$\begin{aligned} \|e^{(k+1)}\|_A^2 &= \langle e^{(k+1)}, e^{(k)} \rangle_A \\ &= \|e^{(k)}\|_A^2 - \alpha_k \langle r^{(k)}, e^{(k)} \rangle_A \\ &= \left(1 - \frac{\langle r^{(k)}, r^{(k)} \rangle^2}{\langle r^{(k)}, r^{(k)} \rangle_A \langle r^{(k)}, r^{(k)} \rangle_{A^{-1}}}\right) \|e^{(k)}\|_A^2. \end{aligned}$$

Next, we establish a useful algebraic inequality:

> **Kantorovich's inequality**
>
> If $\theta_i \geq 0$ and $0 < a \leq x_i \leq b$ for $1 \leq i \leq n$, then
>
> $$\left(\sum_{i=1}^n \theta_i x_i\right)\left(\sum_{i=1}^n \frac{\theta_i}{x_i}\right) \leq \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^n \theta_i\right)^2.$$

Proof. By homogeneity, we may assume that $\sum_i \theta_i = 1$ and $ab = 1$. Since $x \mapsto x + \frac{1}{x}$ is convex on $[a, b]$, we have $x_i + \frac{1}{x_i} \leq a + b$ and hence $\sum_i \theta_i x_i + \sum_i \frac{\theta_i}{x_i} \leq \sum_i \theta_i (a+b) = a + b$. The result then follows from the AM–GM inequality. ∎

Now if the eigenvalues of A are $\lambda_1 \geq \cdots \geq \lambda_n > 0$, then by Kantorovich's inequality,

$$\frac{\langle r^{(k)}, r^{(k)} \rangle^2}{\langle r^{(k)}, r^{(k)} \rangle_A \langle r^{(k)}, r^{(k)} \rangle_{A^{-1}}} \geq \frac{4\lambda_1 \lambda_n}{(\lambda_1 + \lambda_n)^2} = \frac{4\kappa}{(\kappa+1)^2},$$

where κ is the (2-norm) condition number of A. Hence

> $$\|e^{(k)}\|_A \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^k \|e^{(0)}\|_A.$$

Residual projection methods

A **residual projection method** is a projection method where A is invertible and $\mathcal{L} = A\mathcal{K}$.

> If A is invertible and $\mathcal{L} = A\mathcal{K}$, then $\tilde{x}$ uniquely minimizes the norm of the **residual** $b - A\tilde{x}$ over $x^{(0)} + \mathcal{K}$.

Proof. For all $x \in x^{(0)} + \mathcal{K}$, we have $\|b - Ax\|^2 = \|b - A\tilde{x}\|^2 + \|A(\tilde{x} - x)\|^2$ because $A(\tilde{x} - x) \in A\mathcal{K}$ and $b - A\tilde{x} \perp A\mathcal{K}$ according to the Petrov–Galerkin conditions. ∎