Polynomial interpolation

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Lagrange interpolation

Suppose that f is a real-valued function defined on a set of *distinct* **nodes** $\{x_j\}_{j=0}^n \subseteq \mathbb{R}$ and let P_n denote the vector space of real polynomial functions of degree *at most* n. Then there exists a unique $p_n \in P_n$ that *interpolates* f at the nodes in the sense that $p_n(x_j) = f(x_j)$ for each j:

$$p_n = \sum_j f(x_j)\ell_j,$$

where $\ell_j := \prod_{i \neq j} \frac{\bullet - x_i}{x_j - x_i} \in P_n$. Clearly, p_n interpolates f since $\ell_j(x_i) = \delta_{ij}$, and it is unique because if $q_n \in P_n$ also interpolates f (at the same nodes), then $p_n - q_n \in P_n$ has $n+1$ distinct zeroes and must therefore be the zero polynomial. The polynomials $\{\ell_j\}_{j=0}^n$ constitute a basis of P_n called the **Lagrange basis**.

Defining the **barycentric weights** $w_j := \frac{1}{\prod_{i \neq j}(x_j - x_i)}$, which notably depend only on the nodes, we can express p_n as $\omega \sum_j \frac{w_j f(x_j)}{\bullet - x_j}$ (except at the nodes – where the values of p_n are given), where $\omega := \prod_i (\bullet - x_i)$ is the **nodal polynomial**. In particular, taking $f = 1$ implies that $\omega \sum_j \frac{w_j}{\bullet - x_j} = 1$, whence we obtain the **barycentric formula**

$$p_n = \frac{\sum_j \frac{w_j f(x_j)}{\bullet - x_j}}{\sum_j \frac{w_j}{\bullet - x_j}}.$$

Hermite interpolation

More generally, suppose that f is a real-valued function defined on a multiset of nodes $\{x_j^{1+\mu_j}\}_{j=0}^m \subseteq \mathbb{R}$, where x_j has multiplicity $1 + \mu_j$ and f is μ_j times differentiable at x_j. Then there exists a unique polynomial $p_n \in P_n$, where $n = m + \sum_j \mu_j$, that interpolates f at the nodes in the sense that $p_n^{(k)}(x_j) = f^{(k)}(x_j)$ for each j and each $0 \leq k \leq \mu_j$.

Indeed, such a polynomial can be constructed recursively as follows: if $m > 0$, let $p_- \in P_{n-1}$ interpolate f at $\{x_j^{1+\mu_j}\}_{j=0}^m \setminus \{x_m^1\}$ and $p_+ \in P_{n-1}$ interpolate f at $\{x_j^{1+\mu_j}\}_{j=0}^m \setminus \{x_0^1\}$, and define $p_n \in P_n$ as

$$\begin{aligned} p_n &:= \frac{x_m - \bullet}{x_m - x_0} p_- + \frac{\bullet - x_0}{x_m - x_0} p_+ \\ &= p_- + \frac{\bullet - x_0}{x_m - x_0}(p_+ - p_-). \end{aligned}$$

Clearly, $p_n(x_j) = f(x_j)$ for each j since $p_-(x_j) = f(x_j) = p_+(x_j)$ for $0 < j < m$, $p_-(x_0) = f(x_0)$, and $p_+(x_m) = f(x_m)$. Moreover, for $k > 0$, we have

$$p_n^{(k)} = p_-^{(k)} + \frac{\bullet - x_0}{x_m - x_0}(p_+ - p_-)^{(k)} + k \cdot \frac{1}{x_m - x_0}(p_+ - p_-)^{(k-1)},$$

so, similarly, $p_n^{(k)}(x_j) = f^{(k)}(x_j)$ for each j and each $0 < k \leq \mu_j$. In the base case $m = 0$, we take p_n to be the μ_0th degree Taylor polynomial of f centred at x_0. Uniqueness follows as above, counting zeroes with multiplicity.

By specifying appropriate values for f and its derivatives at the nodes, we can construct $h_{jk} \in P_n$ such that $p_n = \sum_j \sum_k f^{(k)}(x_j)h_{jk}$ for all f. These polynomials $\{h_{jk}\}$ constitute a basis of P_n called the **Hermite basis**. For instance, if $\mu_j = 0$ for each j, then $h_{j,0} = \ell_j$ and we recover the Lagrange basis functions; if instead $m = 0$, then $h_{0,k} = \frac{1}{k!}(\bullet - x_0)^k$, which is sometimes called a **Taylor basis** function.

Newton interpolation

Newton interpolation recasts Lagrange/Hermite interpolation in a more explicit basis in which the coefficients of the interpolating polynomial can still be efficiently computed. Let $t_0, \dots, t_n$ be an enumeration of the nodes $\{x_j^{1+\mu_j}\}_{j=0}^m$ (with multiplicity, in any order). The **Newton basis** $\{\omega_j\}_{j=0}^n$ of P_n is then defined as $\omega_j := \prod_{i<j}(\bullet - t_i)$.

To compute the coefficients of the interpolating polynomial p_n in this basis, we define the **divided difference** $f[t_0, \dots, t_n]$ as the $(\bullet)^n$ coefficient of p_n in the **monomial basis** $\{(\bullet)^j\}_{j=0}^n$ (which for brevity we will refer to as the "leading coefficient" despite the fact that it may be zero). The coefficients of p_n in the Newton basis are then successive divided differences of f.

> **Coefficients of interpolating polynomial in Newton basis**
>
> $$p_n = \sum_j f[t_0, \dots, t_j]\omega_j$$

Proof. Write $p_n = \sum_j c_j\omega_j$. For each j, the polynomial $\sum_{k\leq j} c_k\omega_k \in P_j$ interpolates f at $\{t_0, \dots, t_j\}$ since $\omega_k(t_i) = 0$ (with multiplicity) for all $i \leq j < k$. Clearly, its leading coefficient is c_j, so by definition, $c_j = f[t_0, \dots, t_j]$. ∎

From Lagrange interpolation, we obtain an explicit formula for divided differences when the t_j are *distinct* in terms of the barycentric weights:

$$f[t_0, \dots, t_n] = \sum_j w_j f(t_j).$$

More generally, the recursive construction in Hermite interpolation shows that divided differences obey a recurrence relation.

> **Recurrence relation for divided differences**
>
> Suppose that the nodes are ordered such that $t_0 = t_n$ implies that $t_0 = \cdots = t_n$. Then
>
> $$f[t_0, \dots, t_n] = \begin{cases} \dfrac{f[t_1, \dots, t_n] - f[t_0, \dots, t_{n-1}]}{t_n - t_0} & \text{if } t_0 \neq t_n, \\ \dfrac{f^{(n)}(t_0)}{n!} & \text{if } t_0 = t_n. \end{cases}$$

Proof. This follows immediately from the construction above: namely, if $t_0 \neq t_n$, then $p_n = p_- + \frac{\bullet - t_0}{t_n - t_0}(p_+ - p_-)$, where $p_- \in P_{n-1}$ interpolates f at $\{t_0, \dots, t_{n-1}\}$ and $p_+ \in P_{n-1}$ interpolates f at $\{t_1, \dots, t_n\}$; otherwise, if $t_0 = t_n$, then p_n is the nth degree Taylor polynomial of f centred at t_0. ∎

This also yields a recursive algorithm for evaluating p_n known as **Neville's algorithm**. To wit, suppose that the nodes are ordered such that $t_i = t_j$ for $i < j$ implies that $t_i = \cdots = t_j$, and let $p_{i,j} \in P_{j-i}$ interpolate f at $\{t_i, \ldots, t_j\}$ so that $p_n = p_{0,n}$. Then

$$p_{i,j}(t) = \begin{cases} \dfrac{(t-t_i)p_{i+1,j}(t) - (t-t_j)p_{i,j-1}(t)}{t_j - t_i} & \text{if } t_i \neq t_j, \\ \displaystyle\sum_{k=0}^{j-i} \frac{f^{(k)}(t_i)}{k!}(t-t_i)^k & \text{if } t_i = t_j. \end{cases}$$

> **Properties of divided differences**
>
> - (**Linearity**) If $\alpha, \beta \in \mathbb{R}$, then $(\alpha f + \beta g)[t_0, \ldots, t_n] = \alpha \cdot f[t_0, \ldots, t_n] + \beta \cdot g[t_0, \ldots, t_n]$.
> - (**Symmetry**) If σ is a permutation of $\{0, \ldots, n\}$, then $f[t_0, \ldots, t_n] = f[t_{\sigma(0)}, \ldots, t_{\sigma(n)}]$.
> - (**Factor property**) If $n \geq 1$, then $\big((\bullet - t_0)f\big)[t_0, \ldots, t_n] = f[t_1, \ldots, t_n]$.

Proof.

- (Linearity) If $p_f \in P_n$ and $p_g \in P_n$ interpolate f and g, respectively, at $\{t_0, \ldots, t_n\}$, then $\alpha p_f + \beta p_g \in P_n$ interpolates $\alpha f + \beta g$ at $\{t_0, \ldots, t_n\}$.
- (Symmetry) This is immediate since the definition of the divided difference is independent of the ordering of the nodes.
- (Factor property) If $p_f \in P_{n-1}$ interpolates f at $\{t_1, \ldots, t_n\}$, then $(\bullet - t_0)p_f \in P_n$ interpolates $(\bullet - t_0)f$ at $\{t_0, \ldots, t_n\}$. ∎

In fact, the recurrence relation, excluding the base case, can be derived solely from these three properties:

$$\begin{aligned} (t_n - t_0) \cdot f[t_0, \ldots, t_n] &= \big(((\bullet - t_0) - (\bullet - t_n))f\big)[t_0, \ldots, t_n] \\ &= \big((\bullet - t_0)f\big)[t_0, \ldots, t_n] - \big((\bullet - t_n)f\big)[t_0, \ldots, t_n] \\ &= f[t_1, \ldots, t_n] - f[t_0, \ldots, t_{n-1}] \end{aligned}$$

Thus, these properties along with the property $f[t_0, \ldots, t_n] = \frac{f^{(n)}(t_0)}{n!}$ when $t_0 = \cdots = t_n$ characterize divided differences.

The identity $f[t_0, t_1] = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$ for $t_0 \neq t_1$ suggests another relationship between divided differences and derivatives: if, say, $t_0 < t_1$, f is continuous on $[t_0, t_1]$, and f' exists on (t_0, t_1), then the mean value theorem amounts to the assertion that there exists a $\xi \in (t_0, t_1)$ such that $f[t_0, t_1] = f'(\xi)$. This generalizes readily to divided differences and derivatives of higher order.

> **Mean value theorem for divided differences**
>
> Suppose that $a := \min\{t_j\}_{j=0}^n < \max\{t_j\}_{j=0}^n =: b$. If $f^{(n-1)}$ is continuous on $[a, b]$ and $f^{(n)}$ exists on (a, b), then there exists a $\xi \in (a, b)$ such that
>
> $$f[t_0, \ldots, t_n] = \frac{f^{(n)}(\xi)}{n!}.$$

Proof. Let $p_n \in P_n$ interpolate f at $\{t_0, \ldots, t_n\}$. Then $f - p_n$ has $n+1$ zeroes in $[a, b]$ (with multiplicity), so by repeated applications of Rolle's theorem, $(f-p)^{(n)} = f^{(n)} - f[t_0, \ldots, t_n]n!$ has a zero $\xi \in (a, b)$. ∎

As a consequence, we can express the error in polynomial interpolation, which for $t_0 = \cdots = t_n$ reduces to the statement of Taylor's theorem.

> **Polynomial interpolation error**
>
> Suppose that $a := \min \{t_j\}_{j=0}^n \cup \{t\} < \max \{t_j\}_{j=0}^n \cup \{t\} =: b$. If $f^{(n)}$ is continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b), then there exists a $\xi \in (a, b)$ such that
>
> $$f(t) - p_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_j (t - t_j).$$

Proof. Let $p_n \in P_n$ interpolate f at $\{t_0, \ldots, t_n\}$. Then $p_n + f[t_0, \ldots, t_n, t]\omega_{n+1}$ interpolates f at $\{t_0, \ldots, t_n, t\}$, so the result follows from the mean value theorem for divided differences. ∎

We also have the identity $(fg)[t_0, t_1] = f[t_0] \cdot g[t_0, t_1] + f[t_0, t_1] \cdot g[t_1]$, where the case $t_0 = t_1$ is the product rule for derivatives (which also follows from taking $t_1 \to t_0$ in the case $t_0 \neq t_1$). More generally, we have the following identity, which for $t_0 = \cdots = t_n$ reduces to the generalized product rule for derivatives $(fg)^{(n)} = \sum_j \binom{n}{j} f^{(j)} g^{(n-j)}$.

> **Product rule for divided differences**
>
> $$(fg)[t_0, \ldots, t_n] = \sum_j f[t_0, \ldots, t_j] \cdot g[t_j, \ldots, t_n]$$

Proof. Let $p_n \in P_n$ interpolate f at $\{t_0, \ldots, t_n\}$. Then $(fg)[t_0, \ldots, t_n] = (p_n g)[t_0, \ldots, t_n]$ since fg agrees with $p_n g$ on $\{t_0, \ldots, t_n\}$. By linearity and the factor property, we have

$$\begin{aligned}
(p_n g)[t_0, \ldots, t_n] &= \left(\sum_j f[t_0, \ldots, t_j]\omega_j g\right)[t_0, \ldots, t_n] \\
&= \sum_j f[t_0, \ldots, t_j] \cdot (\omega_j g)[t_0, \ldots, t_n] \\
&= \sum_j f[t_0, \ldots, t_j] \cdot g[t_j, \ldots, t_n]. \qquad \blacksquare
\end{aligned}$$

Furthermore, from the recurrence relation for divided differences, we see that if $f \in C^0$, then $f[t_0, \ldots, t_n]$ is jointly continuous in $t_0, \ldots, t_n$ wherever they are distinct; if $f \in C^n$, the mean value theorem for divided differences implies that it is jointly continuous everywhere.