A proof of Zarantonello’s lemma

Note: The following lemma appears in a number of texts by Saad [4, 5], who refers to Rivlin [3] for its proof. However, Rivlin’s book does not appear to contain this lemma – at least, not explicitly in the form presented by Saad. A search for this lemma in other publications [1, 2] reveals that it was originally introduced in a slightly different form by Varga [6], who in turn attributes it to a personal communication with Zarantonello and provides a rather complicated proof of the lemma. What follows is a simple proof of the lemma as it appears in Saad’s books.

Lemma (Zarantonello). Let \( r > 0 \) and \( \gamma \in \mathbb{C} \) be a point not enclosed by \( C_r = \{ z \in \mathbb{C} : |z| = r \} \) (i.e., \( |\gamma| \geq r \)). Then

\[
\min_{p \in \Pi_k, p(\gamma) = 1} \max_{z \in C_r} |p(z)| = \left( \frac{r}{|\gamma|} \right)^k,
\]

where \( \Pi_k \) denotes the set of polynomials of degree at most \( k \), with the minimum attained by \( p(z) = (z/\gamma)^k \).

Proof. We will prove that if \( p \in \Pi_k \) and \( M := \max_{z \in C_r} |p(z)| \), then for all \( |z| \geq r \),

\[
|p(z)| \leq \frac{M}{r^k} |z|^k,
\]

from which the lemma will follow by setting \( z = \gamma \) and taking the infimum over all such \( p \) with \( p(\gamma) = 1 \). Let \( \tilde{p} \) be the polynomial defined by \( \tilde{p}(w) = w^k p(r^2/w) \). For \( |w| \leq r \), we have

\[
|\tilde{p}(w)| \leq \max_{w \in C_r} |\tilde{p}(w)| = r^k \max_{w \in C_r} |p(r^2/w)| = Mr^k,
\]

where the inequality follows from the maximum modulus principle. Now if \( |z| \geq r \), then \( w := r^2/z \) satisfies \( |w| \leq r \), so \( |\tilde{p}(w)| \leq Mr^k \). But \( |\tilde{p}(w)| = (r^2/|z|)^k |p(z)| \), whence \( |p(z)| \leq M |z|^k / r^k \). □
References


