Derivative tests

Nicholas Hu · Last updated on 2025-11-27

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. We denote the interior of I by I°.

Monotonicity

We say that f is **increasing** if $f(x) \leq f(y)$ for all $x, y \in I$ with $x < y$ and that f is **strictly increasing** if the former inequality is strict. We define **decreasing** and **strictly decreasing** analogously.

> **Characterization of monotonicity**
>
> Suppose that f is continuous on I and differentiable on I°.[1] Then f is increasing on I if and only if $f' \geq 0$ on I°. In addition, f is strictly increasing on I if $f' > 0$ on I°.[2]

Extremality

We say that $x \in I$ is a **(global) minimizer** and that the value $f(x)$ is a **(global) minimum** of f if $f(x) \leq f(y)$ for all $y \in I$. We define **(global) maximizer** and **(global) maximum** analogously.

> **Extreme value theorem**
>
> If I is closed and bounded and f is continuous on I, then f has a minimizer and a maximizer in I.

We say that $x \in I$ is a **local minimizer** and that the value $f(x)$ is a **local minimum** of f if there exists an $\varepsilon > 0$ such that $f(x) \leq f(y)$ for all $y \in I \cap (x - \varepsilon, x + \varepsilon)$. We define **local maximizer** and **local maximum** analogously. Clearly, any global extremizer must also be a local extremizer.

> **Fermat's theorem**
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> Suppose that $x \in I^\circ$ is a local extremizer of f. If f is differentiable at x, then $f'(x) = 0$ (that is, x is a **stationary point** of f).

Thus, under the hypotheses of the extreme value theorem, any extremizer of f will either be an interior point of I and hence a point at which f is stationary *or* not differentiable, or else will be a boundary point of I. We call a point of the former type (that is, a point at which f is stationary *or* not differentiable) a **critical point** of f.

> **First derivative test**
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> If f is continuous at x and there exists an $\varepsilon > 0$ such that $f' \leq 0$ on $(x - \varepsilon, x)$ and $f' \geq 0$ on $(x, x + \varepsilon)$, then x is a local minimizer of f.

> **Second derivative test**
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> If f is stationary at x and $f''(x) > 0$, then x is a local minimizer of f.

Convexity

We say that f is **convex** if $f((1-\theta)x+\theta y) \leq (1-\theta)f(x)+\theta f(y)$ for all *distinct* $x, y \in I$ and $\theta \in (0,1)$ and that f is **strictly convex** if the inequality is strict. We define **concave** and **strictly concave** analogously.

> **Secant and tangent line characterizations of convexity**
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> Given an $x \in I$, let $g(y;x) := \frac{f(y)-f(x)}{y-x}$ for all $y \in I \setminus \{x\}$. Then f is convex (resp., strictly convex) on I if and only if g is increasing (resp., strictly increasing) in y for all $x \in I$. In addition, if f is differentiable on I°, then f is convex (resp., strictly convex) on I if and only if $g(y;x) \geq f'(x)$ (resp., $g(y;x) > f'(x)$) for all $x \in I^\circ$ and $y \in I \setminus \{x\}$.

> **First-order characterization of convexity**
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> Suppose that f is continuous on I and differentiable on I°. Then f is convex (resp., strictly convex) on I if and only if f' is increasing (resp., strictly increasing) on I°.

> **Second-order characterization of convexity**
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> Suppose that f is continuous on I and twice differentiable on I°. Then f is convex on I if and only if $f'' \geq 0$ on I°. In addition, f is strictly convex on I if $f'' > 0$ on I°.

> **Minimizers of convex functions**
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> If f is convex and x is a local minimizer of f, then x is a global minimizer of f. In addition, if f is strictly convex, then f has *at most one* local minimizer.

We say that the graph of f has an **inflection point** at $(x, f(x))$ if it has a tangent line at $(x, f(x))$ (which may be vertical) and its **(signed) curvature** $\kappa := \frac{f''}{[1+(f')^2]^{3/2}}$ *changes sign* at x (though κ need not exist at x itself). At such a point, f changes from strictly convex to strictly concave or vice-versa.

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1. If I is open, then $I^\circ = I$ and the hypothesis reduces to f being differentiable on I. ↩

2. The converse is false in general: consider $f(x) = x^3$ on $I = \mathbb{R}$. ↩