Worksheet 6

July 3, 2019

1. Consider the poset $(\mathbb{N}, \leq)$ as a category. Compute $\hom(m, n)$ for each n and m.

2. For each natural numer n, we can consider the category $\mathbf{n}$ whose objects are the numbers $1, \ldots, n$ ordered in the usual way. Diagrammatically, we have:

$$\mathbf{1} = *$$

$$\mathbf{2} = * \longrightarrow *$$

$$\mathbf{3} = * \longrightarrow * \longrightarrow *$$

   etc. Describe what it means to give a functor from $\mathbf{1}$, $\mathbf{2}$, or $\mathbf{3}$ to a category C.

3. (a) Show that the data of a functor between posets is the same as the data of an order-preserving set map.

   (b) Define Poset to be the category of partially ordered sets with order-preserving maps between them. As discussed in class, we can consider a category Cat of *small* categories whose objects are categories and morphisms are functors between them.

   Show that the association of a category to a poset defines a functor Poset $\to$ Cat. (Moreover, this is an "embedding" of categories. See problem on the Yoneda lemma to make this precise.)

4. Let $S, T\colon C \to P$ be two parallel functors from an arbitrary category C to a poset P (viewed as a category). Show that there is a natural transformation $S \implies T$ if and only if for all $c \in C$, $Sc \leq Tc$.

5. Givens small categories $\mathcal{A}$ and $\mathcal{B}$, we can consider a category $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ of functors from $\mathcal{A}$ to $\mathcal{B}$. Shockingly, the objects are functors from $\mathcal{A}$ to $\mathcal{B}$; the morphsims are natural transformations between them. Morphsims sets in this category are often written as $\mathrm{Nat}(F, G)$ (for F, G functors).

   The Yoneda lemma states the following: given a contravariant functor from a category C to the category of sets and set maps, i.e. $F\colon C^{\mathrm{op}} \to \mathrm{Set}$, and an object $b \in C$, there is bijection

   $$\mathrm{Nat}(\hom(-, b), F) \simeq F(b)$$

   which is natural in b and F.

   (a) Establish the bijection given in the lemma.

   (b) A functor $G\colon C \to \mathcal{D}$ is said to be **fully faithful** if, for each $a, b \in C$, G establishes a bijection $\hom_C(a, b) \simeq \hom_{\mathcal{D}}(F(a), F(b))$. Show that, for C a small category, the assignment

   $$c \mapsto \hom_C(-, c)$$

   corresponds to a fully faithful functor

   $$y\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}).$$

   REMARK 1. *We call the category* $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ *the category of presheaves in C. So the Yoneda lemma establishes an "inclusion" of any small category C into its category of presheaves.*

6. A **discrete category** is one where all morphisms are identities.

   (a) Show that we have a functor $\mathrm{Set} \to \mathrm{Cat}$ given by sending a set to the discrete category with (small) object collection given by the set, and only identity morphisms.

   (b) Is this functor fully faithful? (See problem 5(b) for relevant definition.)

7. A **groupoid** is a category where every morphism is invertible.

   (a) Give an example of a groupoid.

(b) Given a topological space X, one can define the **fundamental gropoid of** X as the category with object the points of X and morphisms all paths in X. We denote this by $\Pi_{\leq 1}X$. Show that $\Pi_{\leq 1}X$ is indeed a groupoid.

8. Write down the universal property of the coproduct in the category Grp of (not necessarily abelian) groups. Compute the coproduct of the groups Z with itself. Compare this to the coproduct in the category Ab of abelian groups.

9. Given a group G, we can define a category BG with one object $*$ and with morphisms given by the elements of G. Composition is given by multiplication in G.

   (a) What is the identity morphisms from $*$ to $*$ in BG?

   (b) Given G, H groups, check that $B(G \times H) = BG \times BH$, where on one side we have the cartesian product of groups (i.e. the usual group structure on the cartesian product of the underlying sets, not necessarily the product in Grp) and on the other we have the product in the category of small categories.

10. A arrow $f\colon b \to c$ in a category C is set to be an **epimorphism** if, for all $g, g'\colon c \to d$ in C,

$$g'f = gf$$

implies

$$g = g'.$$

Dually, f is said to be **monic** or a **monomorphism** if, for all $h, h'\colon a \to b$ in C,

$$fh = fh'$$

implies

$$h = h'.$$

   (a) Determine the epi- and mono- morphisms in the category Set .

   (b) Show that, in the category Ab, not every every epimorphism is surjective.

   (c) Show that a $f \in \operatorname{mor} C$ is epi if and only if it is mono when viewed as a morphism in C^{op}.

11. **(Challenging if you haven't worked with adjunctions.)** Given functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$, we say that F **is left adjoint to** G **(and** G **is right adjoint to** F**)**, and write $F \dashv G$, if there are isomorphisms

$$\hom_{\mathcal{C}}(A, GB) \simeq \hom_{\mathcal{D}}(FA, B)$$

natural in $A \in \mathcal{C}$ and $B \in \mathcal{D}$.

(a) Show that, given $F \dashv G$ as defined above, we obtain natural transformations

$$\epsilon: 1_{\mathcal{C}} \to GF$$

and

$$\eta: FG \to 1_{\mathcal{D}}$$

with components defined by taking $B = FA$ and $A = GB$ in the above. (These are called the **unit** and **counit** of the adjunction.)

(b) Given two functors F and G as above (but not necessarily an adjoint pair), and $\epsilon: 1 \to GF$, $\eta: FG \to 1$, come up with necessary and sufficient conditions for ϵ and η to arise from an adjunction.

(c) Conclude that the data of natural isomorphisms

$$\hom_{\mathcal{C}}(A, GB) \simeq \hom_{\mathcal{D}}(FA, B)$$

are equivalent to that of natural transformations ϵ and η satisfying the condition you found in (b).