Worksheet 5

Problem 1. Euler characteristic from critical points (surfaces in $\mathbf{R}^3$).

Let Σ be a connected, oriented, compact surface without boundary embedded in $\mathbf{R}^3$. Consider the height function $z\colon \Sigma \to \mathbf{R}$ given by projection on the z-axis.

(a) Check that the critical points on Σ with respect to the function z are precisely the points on Σ whose tangent planes are parallel to the xy-plane. As a reminder, a point $p \in \Sigma$ is a critical point if the derivative of z at p is zero. The derivative of z at p is a function which associates to each tangent vector to Σ at p the z component of the tangent vector.

A result from *Morse theory*[1] says that after a small isotopy of Σ, we may assume that the critical points are isolated, and that near each critical point, the surface looks like a minimum, a saddle, or a maximum (with respect to the height function). One can also assume after a small isotopy that all the critical points have different critical values (occur at different heights). Consider cross-sections of the surface Σ (intersections of Σ with planes parallel to the xy-plane). Starting with a plane meeting the z-axis at a very large negative value so that its intersection with Σ is empty (Σ is compact), we consider how the cross-section changes as we move the plane upwards and "scan" the surface from bottom to top. A generic cross-section (i.e. one not containing a critical point) looks like a union of circles in the plane. Recall that a circle (in a union of circles in the plane) is called *innermost* if its "inside", i.e. the disc that it bounds in the plane, does not contain any other circle.

(b) Convince yourself that when passing a minimum, an innermost circle appears in the cross-section. In particular, a point first appears at the critical value, and then becomes a very small circle centered at the point. Similarly, when passing a maximum, an innermost circle disappears. Convince yourself that when passing a saddle point, either two circles merge or one circle splits. Finally, when passing through a region with no critical points, the cirlces in the plane simply undergo a planar isotopy (i.e. they move around in the plane but undergo no change in topology).

(c) Find (draw) an embedding of the sphere S^2 into $\mathbf{R}^3$ so that with respect to the height function, all three types of critical points exist. Draw the "movie" of cross-sections.

We have been considering level sets of the height function, i.e. for each $\alpha \in \mathbf{R}$, we have the set of points in Σ whose z-coordinate is α. Now consider *sublevel sets*: for each $\alpha \in \mathbf{R}$, the set of points in Σ whose z-coordinate is less than or equal to α. Consider how the sublevel sets change as α increases. A generic sublevel set is a surface with boundary, the boundary at the top of the sublevel set. A disc appears when passing a minimum, and a disc disappears when passing a maximum.

(d) Convince yourself that a band appears in the sublevel set when passing a saddle. The appearance of a band is the appearance of a rectangle in the plane with two opposite edges lying on the boundary of the existing sublevel set. Check that band attachment changes the boundary of the sublevel set by merging two components or spliting one component.

[1]One can find proofs of all Morse-theoretic claims in this problem in Milner's book *Morse Theory*.

(e) Show that the Euler characteristic of the surface is the number of maxima plus the number of minima minus the number of saddles.

Problem 2. Let Σ be a connected oriented compact surface with boundary. Consider attaching a band to Σ (i.e. choose two intervals contained in the boundary of Σ, and identify them with two opposite edges of a rectangle). Note that the boundary components of Σ inherit orientations coming from the orientation of Σ, as does the boundary of the rectangle after orienting the rectangle. Show that if the band is attached in a way preserving the orientations on the boundaries, then the resulting surface after band attachment is orientable. Show that if the band is attached in a way that does not respect the orientations of the boundaries, then the resulting surface is non-orientable.

Problem 3. The Morse theory set up of Problem 1 works for surfaces embedded in $\mathbf{R}^4$. Instead of using the height function z, we will use the time function t, where the coordinates on $\mathbf{R}^4$ are (x, y, z, t). The level sets with respect to time are copies of $\mathbf{R}^3$. If Σ a connected compact surface without boundary embedded in $\mathbf{R}^4$, then after a small isotopy, we may assume that the critical points with respect to the time function are isolated, occur at different times, and the surface locally looks like either a minimum, saddle, or maximum. We will visualize the surface through its collection of cross-sections, thought of as a "movie."

Generically, a cross-section of Σ looks like a union of circles in $\mathbf{R}^3$ (i.e. a link). During a portion of the movie when no critical points occur, the link undergoes an isotopy. When passing a minimum, a small unknot appears. Just like in one dimension lower, it is "innermost" in the sense that it bounds a disc disjoint from the rest of the link. When passing a maximum, a small unknot which bounds a disc disjoint from the rest of the link disappears. When passing a saddle, a band move occurs (an embedded rectangle appears for just one moment with two opposite edges lying on the link, and immediately after the interior of the rectangle along with the two edges on the link disappear, leaving behind the remaining two edges). Note that the band may look complicated with respect to the link.

(a) Consider the following movie. First there is nothing in $\mathbf{R}^3$. Then a "birth" occurs producing an unknot. Then a band move happens in such a way that the unknot is left after the band move (in particular, note that a general band move no longer must split or merge if it is not required to respect orientation). Then the unknot disappears in a "death." Show that resulting connected compact boundaryless surface has Euler characteristic 1 and hence is homeomorphic to $\mathbf{RP}^2$. In particular, we have an explicit embedding of $\mathbf{RP}^2$ into $\mathbf{R}^4$.

When describing a movie of an embedded surface in $\mathbf{R}^4$, if we orient all links appearing in the movie and require that the bands attach in such a way that respects the orientations, then the surface in $\mathbf{R}^4$ will be oriented.

(b) Recall that Seifert's algorithm is a construction of a Seifert surface for a knot K given a diagram for K. It starts with some number s of discs and then attaches some number c of bands to those discs to create the Seifert surface. Consider the movie which first starts with s births producing the unlink, then c band attachments in exactly the way they are attached in Seifert's algorithm. We now have the knot K. Then reverse the band attachments by attaching new bands (these new bands have the same underlying rectangles as the old bands, but now they go from the new set of edges which are a subset of K to the old set of edges which are a subset of the unlink). Finally, there are s deaths. We have obtained a compact connected oriented surface without boundary in $\mathbf{R}^4$, one of whose cross-sections is K (so every knot appears as the cross-section of a surface in $\mathbf{R}^4$). What is the genus of this surface in terms of s and c?

(c) Can you find a nontrivial knot which is a cross-section of an embedded sphere?

Problem 4. Let K be a knot in $\mathbf{R}^3 \times 0$, thought of as the subset $\mathbf{R}^3 \times 0 \subset \mathbf{R}^3 \times [0, \infty) \subset \mathbf{R}^4$. Let Σ be a connected, oriented, compact surface embedded in $\mathbf{R}^3 \times [0, \infty)$ with boundary equal to the knot K in $\mathbf{R}^3 \times 0$. The minimal genus of such a surface Σ is called the *slice genus* or the *4-ball genus* of K, and is denoted $g_4(K)$.

(a) Show that you can achieve a crossing change of a knot via two band moves. In other words, if K' is obtained from K via a crossing change, show that K' can be obtained from K via two band moves.

(b) Show that $g_4(K) \leq u(K)$ where $u(K)$ is the unknotting number of K.

Problem 5. A knot in $\mathbf{R}^3$ is in *bridge position* if with respect to the height function all maxima occur at the same height and all minima occur at the same height. Every knot may be put in bridge position.

(a) Draw diagrams of the trefoil and figure eight in bridge position.

(b) Show that the number of maxima is the same as the number of minima.

The *bridge number* $\mathrm{br}(K)$ of a knot K is the least number $n \geq 1$ for which it is possible to put K in bridge position with n maxima.

(c) Show that the only knot with bridge number 1 is the unknot.

Given a bridge diagram for a knot K, draw a line parallel to the line containing the minima beneath the minima so that the entire diagram lies above the line. Reflect the diagram over the line.

(d) If n is the number of minima in the diagram for K, do n band moves, each of which joining a minima of K with a maxima of the mirror of K, in such a way that the band lies in the plane and crosses the line once. Show that resulting link is the unlink.

(e) Observe that an orientation of K determines one of $m(K)$. Let $-m(K)$ denote the $m(K)$ equipped with the opposite orientation. The band moves in part (d) respect these orientations. The first band move creates $K\#(-m(K))$. Show that the rest of the band moves, together with n deaths, one for each component of the resulting unlink, yields a movie for an embedded disc in $\mathbf{R}^3 \times [0, \infty)$ with boundary $K\#(-m(K))$. In particular, $g_4(K\#(-m(K))) = 0$. Can you now do part (c) of Problem 3?