# Worksheet 2

**Problem 1.** Euler characteristic of surfaces.

(a) (Gluing) Let $\Sigma'$ and $\Sigma''$ be connected compact surfaces, each with nonempty boundary. Choose a boundary component of $\Sigma'$ and a boundary component of $\Sigma''$ and identify them in any way to obtain a new connected compact surface $\Sigma = \Sigma' \cup_{S^1} \Sigma''$. Draw a picture of an example of this. Show that $\chi(\Sigma) = \chi(\Sigma') + \chi(\Sigma'')$.

(b) (Self-Gluing) Let $\Sigma$ be a connected compact surface, with at least two boundary components. Choose two boundary components of $\Sigma$ and identify them in any way to obtain $\Sigma'$. Draw a picture of an example of this. Show that $\chi(\Sigma') = \chi(\Sigma)$. If $\Sigma$ is an annulus, which surfaces $\Sigma'$ can you obtain from this construction? *Hint:* There's more than one.

(c) (Deleting a disc) Let $\Sigma$ be a connected compact surface. Choose a subset $D \subset \mathrm{Int}(\Sigma)$ of the interior of $\Sigma$ that homeomorphic to a closed disc, and let $\Sigma' = \Sigma \setminus \mathrm{Int}(D)$ be the complement of the interior of $D$. Observe that $\Sigma'$ is a connected compact surface. Show that $\chi(\Sigma') = \chi(\Sigma) - 1$. *Hint:* Use part (a).

(d) (Connected Sum) Let $\Sigma'$ and $\Sigma''$ be connected compact surfaces. Recall that the connected sum $\Sigma = \Sigma' \# \Sigma''$ is obtained by deleting discs from each of $\Sigma'$ and $\Sigma''$ and identifying the resulting boundary components. Show that $\chi(\Sigma) = \chi(\Sigma') + \chi(\Sigma'') - 2$. *Hint:* Use parts (a) and (c).

*Remark.* It's often helpful to try to come up with a short phrase which conceptually captures the result of lemma or problem. Can you come up with short phrases for each of the previous results? For example, I think of part (c) as "deleting a disc drops Euler characteristic by 1."

**Problem 2.** Let $\Sigma_3$ be the closed oriented surface of genus 3 (recall that a manifold is called closed if it is compact and has no boundary).

(a) Delete six discs from $S^2$ to obtain $\Sigma$. Show that $\Sigma_3$ can be obtained from $\Sigma$ by using the self-gluing move of Problem 1 part (b) three times. Use part (c) and the fact that $\chi(S^2) = 2$ to compute $\chi(\Sigma)$, and use part (b) to compute $\chi(\Sigma_3)$.

(b) Show that the Euler characteristic of the torus is zero using the relations established in Problem 1. Show that $\Sigma_3$ can be obtained by taking the connect sum of three tori. Use part (d) to compute $\chi(\Sigma_3)$.

**Problem 3.** The *pair of pants* is the connected compact surface obtained by deleting three discs from $S^2$ (why is it called the pair of pants?). A *pair of pants decomposition* of an oriented surface $\Sigma$ is a collection of simple closed curves on $\Sigma$ with the property that by cutting the surface along the curves one obtains a collection of pairs of pants. Two decompositions are considered the same if there is a homeomorphism taking the first set of curves to the second.

(a) Show that every connected closed oriented surface of genus $g \geq 2$ admits a pair of pants decomposition.

(b) Show that there must be exactly four pairs of pants in any pants decomposition of $\Sigma_3$. *Hint:* use Euler characteristic.
(c) Show that the torus and sphere do not admit pairs of pants decompositions.

*Remark.* It turns out there are exactly two pairs of pants decompositions of the closed oriented surface of genus 2. Can you find them? How many pairs of pants decompositions of $\Sigma_3$ can you find? Pairs of pants decompositions of closed oriented surfaces are in one-to-one correspondence with finite 3-regular graphs (allowing self-loops and multiple edges) and can be used to construct hyperbolic structures on surfaces.

**Problem 4.** Choose a diagram for the figure-eight. Apply Seifert's algorithm to the diagram to construct a Seifert surface for the figure-eight. Compute its genus.

**Problem 5.** Show that the genus of the $(2, q)$ torus knot (for $q \geq 3$ odd) is at most $(q-1)/2$.

*Remark.* Compare this with Problem 6 of Worksheet 1.

**Problem 6.** Let

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

be a chain complex of finite-dimensional vector spaces. Explicitly, each $C_n$ is a finite-dimensional vector space, each $\partial_n \colon C_n \to C_{n-1}$ is a linear map, and each composite $\partial_n \circ \partial_{n+1}$ is the zero map. Furthermore, assume that there are only finitely many nonzero $C_n$. The homology groups $H_n$ for $n \geq 0$ are defined to be the quotient vector spaces $H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

Show that

$$\sum_{n=0}^{\infty} (-1)^n \dim C_n = \sum_{n=0}^{\infty} (-1)^n \dim H_n.$$

This integer is referred to as the Euler characteristic of the chain complex, and the above equality is often phrased as "Euler characteristic can be computed at either the chain level or at the level of homology."

**Problem 7.** The first homology group $H_1(X)$ of a path-connected space $X$ is isomorphic to the abelianization of its fundamental group $\pi_1(X)$. Concretely, if we have a presentation for $\pi_1(X)$, we obtain a presentation for $H_1(X)$ by adding the commutator $aba^{-1}b^{-1}$ as a relation for each pair of generators $a, b$ in the presentation. This forces every pair of elements to commute.

Using the Wirtinger presentation, show that the first homology group of the complement of a knot is $\mathbf{Z}$. What is the first homology group of the complement of a link with $\ell$ components?

**Problem 8. Completely optional**. Euler characteristic of 3-manifolds. Let $M$ be a closed oriented connected 3-manifold. We will see that $\chi(M) = 0$ in this exercise, citing various results in algebraic topology.

*Poincaré duality* (PD) implies that the following homology and cohomology groups of $M$ are isomorphic:

$$
\begin{tikzcd}
H_3(M) \arrow[rrddd, leftrightarrow] & & H^3(M;\mathbf{Z}) \\
H_2(M) \arrow[rrd, leftrightarrow] & & H^2(M;\mathbf{Z}) \\
H_1(M) \arrow[rru, leftrightarrow] & & H^1(M;\mathbf{Z}) \\
H_0(M) \arrow[rruuu, leftrightarrow] & & H^0(M;\mathbf{Z})
\end{tikzcd}
$$

or explicitly

$$H_0(M) \cong H^3(M;\mathbf{Z}) \qquad H_1(M) \cong H^2(M;\mathbf{Z}) \qquad H_2(M) \cong H^1(M;\mathbf{Z}) \qquad H_3(M) \cong H^0(M;\mathbf{Z}).$$

(a) Using the fact that $H_0(M) \cong \mathbf{Z}^k \cong H^0(M;\mathbf{Z})$ where $k$ is the number of components of $M$, compute as many of the homology and cohomology groups of $M$ as you can. *Hint:* You should be able to deduce the isomorphism type of four of the above eight groups.

Each of $H_n(M)$ and $H^n(M;\mathbf{Z})$ is a finitely generated abelian group. Let $A$ be any finitely generated abelian group. Let $\mathrm{tor}(A)$ denote the subgroup of elements $a \in A$ for which there exists a positive integer $n$ for which $n \cdot a = a + a + \cdots + a = 0$ (where the group operation in $A$ is written additively). The subgroup $\mathrm{tor}(A)$ is the *torsion subgroup* of $A$ and it is a finite abelian group.

(b) Show that $\mathrm{tor}(A)$ is a subgroup.

The *structure theorem of finitely generated abelian groups* implies that $A$ is isomorphic to $\mathbf{Z}^k \oplus \mathrm{tor}(A)$ for some nonnegative integer $k$. The integer $k$ is called the *rank* of $A$, denoted $\mathrm{rk}(A)$.

The *Universal Coefficients Theorem* (UCT) implies that for all $m \geq 0$, the rank of $H^m(M;\mathbf{Z})$ is equal to the rank of $H_m(M)$ and that the torsion subgroup of $H^m(M;\mathbf{Z})$ is isomorphic to the torsion subgroup of $H_{m-1}(M)$.

(c) Show that $H^1(M;\mathbf{Z})$ has trivial torsion subgroup.

(d) Show that $H_2(M)$ has trivial torsion subgroup (use PD, but also show this using part (a) and UCT).

If you know the homology groups of $M$, then UCT says that you can obtain the isomorphism types of the cohomology groups in the following way: the *free parts*, i.e. the $\mathbf{Z}^k$ parts, are the same as the homology groups, but you push the torsion parts up one degree. PD then gives large constraints on the possibilities of what the original homology groups can be. If you know partial information about the homology and cohomology groups, you can often figure out all of them using PD and UCT. A friend jokingly refers to this process as "sudoku".

(e) Suppose $H_1(M) \cong \mathbf{Z}^k \oplus T$ where $T$ is a finite abelian group, and recall that $M$ is connected. Compute all of the homology and cohomology groups of $M$ using PD and UCT.

*Remark.* Since $H_1(M)$ is the abelianization of $\pi_1(M)$, the fundamental group $\pi_1(M)$ determines all of the homology and cohomology groups of $M$. There are many 3-manifolds with the same homology and cohomology groups, but it is a deep theorem in low-dimensional topology, that I won't state here precisely, that the fundamental group is *pretty much* a perfect invariant of closed connected oriented 3-manifolds.

Recall that the Euler characteristic can be computed as

$$\chi(M) = \sum_{m=0}^{3} (-1)^m \operatorname{rk}(H_m(M)) = \operatorname{rk}(H_0) - \operatorname{rk}(H_1) + \operatorname{rk}(H_2) - \operatorname{rk}(H_3).$$

(f) Show that $\chi(M) = 0$.

*Remark.* Although Euler characteristic is a great invariant of surfaces (it is a perfect invariant of connected closed oriented 2-manifolds, and satisfies the various relations of problem 1), it is a poor invariant of connected closed oriented 3-manifolds.

(g) Let $W$ be a connected closed oriented $n$-manifold. UCT still holds true, as does the modification of PD to $H_k(W) \cong H^{n-k}(W;\mathbf{Z})$ for $k \geq 0$. Show that if $n$ is odd, then $\chi(W) = 0$. If $n = 4$, does the fundamental group of $W$ still determine all of the homology and cohomology groups? If $n = 4$ and the fundamental group is trivial, what can you say about the homology and cohomology groups of $W$?