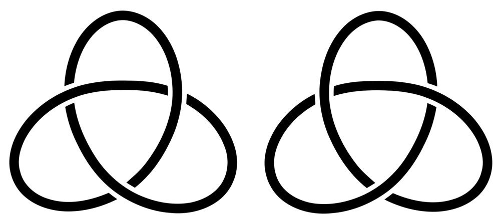
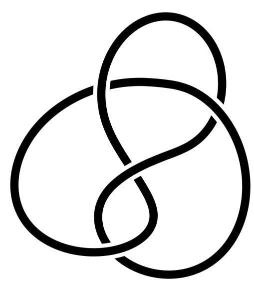
Worksheet 1

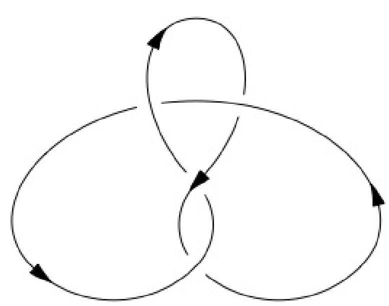
Problem 1. The *mirror* of a knot K, denoted $m(K)$ (or $\overline{K}$ or $-K$), is the knot obtained by reflecting through a plane in 3-space (if you were holding a knot K in your hand, you'd see $m(K)$ when looking at it in a mirror). Given a diagram of K, a diagram for $m(K)$ is obtained by changing every crossing.



The first knot (on the left) is called the *right-handed trefoil* and its mirror (on the right) is called the *left-handed trefoil*. It turns out that these knots are not isotopic. Show that the knot below, called the *figure-eight*, is isotopic to its mirror.



Problem 2. An *oriented* knot is a knot equipped with a direction of travel along the knot. Typically it is drawn with an arrow. Here's the figure-eight equipped with an orientation.

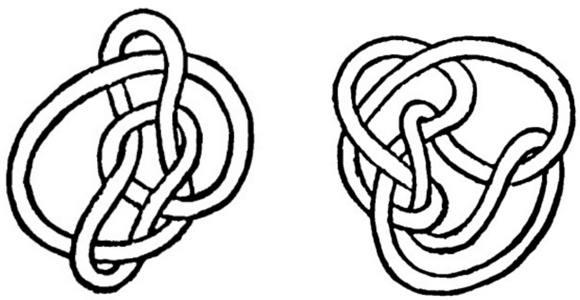


Two oriented knots are called isotopic if there is an isotopy which takes one knot to the other so that the orientations match up (so that the arrows point in the same direction). The *reverse* of an oriented knot is the same knot but equipped with the opposite orientation (draw the arrows in the opposite direction). Show that the trefoils and the figure-eight are isotopic to their reverses.

Remark. The lowest crossing prime knot which is not isotopic to its reverse is 8_{17} (see the remark after Problem 3 for an explanation of this terminology).

Remark. The concepts of mirroring (Problem 1) and an orientation on a knot (Problem 2) are important. Doing the visual manipulations requested in these problems is less important.

Problem 3. Show that the following two knots are isotopic:



Remark. These two knots were originally thought to be distinct. A old (but good) textbook in knot theory called "Knots and Links" by Dale Rolfsen includes a table of prime knots which has become known as "Rolfsen's knot table." It enumerates prime knots of a given crossing number which have now become their standard names, e.g. the trefoil is 3_1, the figure eight is 4_1, and there are two knots with crossing number five called 5_1 and 5_2. *Warning:* Rolfsen's knot table does not distinguish between mirrors and reverses, even when they are not isotopic. The knot drawn above is prime and has crossing number 10 but is mistakenly listed twice as both 10_{161} and 10_{162}. It's now known as the Perko pair (but really it's just one knot).

Remark. This problem is included mostly just for mathematical culture; have some fun playing around with this knot but don't worry too much about actually finding an isotopy between them. If you'd like, you can find an isotopy between them in a Math Stack Exchange post after a quick search.

Problem 4. Determine which of the prime knots on Rolfsen's knot table of crossing number 7 or fewer are tricolorable (there are 15 such knots on the table).

As warned before, Rolfsen's table does not distinguish between mirrors and reverses even when they are distinct. Check for yourself that an oriented knot is tricolorable if and only if its reverse is if and only if its mirror is if and only if its reverse mirror is.

Remark. Many properties if true for an oriented knot will also be true for its mirror, its reverse, and its reverse mirror. This is why the distinction between them is often hazy.

Problem 5. Prove that the *number* of tricolorings is an invariant of the knot.

Hint: Associated to each knot diagram is a set of tricolorings. Given two diagrams that differ by a Reidemeister move, and establish a bijection between their sets of tricolorings.

Remark. You can permute the colors of a given tricoloring to produce new tricolorings. Since a tricoloring by definition must have all three colors, each of the 6 permutations of the colors produces a distinct tricoloring. Hence the number of tricolorings is divisble by 6 (and the set of tricolorings can be naturally partitioned into sets of 6). Can you find a knot with more than 6 tricolorings? *Hint:* There's an example of such a knot with 6 crossings but it's not on Rolfsen's table.

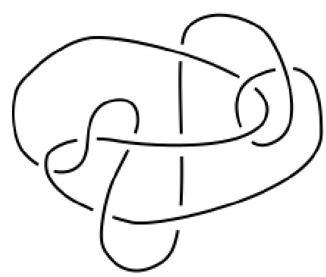
Problem 6. Show that the $(2, q)$ torus knot, where $q \geq 3$ is odd, can be unknotted in $(q-1)/2$ moves. For example, the trefoil can be unknotted in 1 move, and the knot 5_1 can be unknotted in 2 moves. Do you think it can be unknotted in fewer moves? What about the $(3, q)$ torus knots, where 3 and q are coprime? What about the (p, q) torus knots for p, q coprime?

Remark. Establishing upper bounds for unknotting number involves simply finding an unknotting sequence. Finding lower bounds is often much more difficult, and it's where interesting knot invariants come into play. Prof Kronheimer here at Harvard along with his collaborator Prof Mrowka at MIT managed to compute the unknotting numbers of the torus knots in full generality in the early 1990s using gauge theory. In 2004, Jacob Rasmussen, who did his PhD at Harvard under Prof Kronheimer, gave a more elementary proof using Khovanov homology.

Problem 7.

(a) Compute the Wirtinger presentation for the fundamental group of the unknot using a diagram with 3 crossings, and show that the group is indeed a presentation for $\mathbf{Z}$.

(b) Using the following diagram, compute the Wirtinger presentation for the fundamental group of the complement of the knot.



Using visual manipulations, verify that this is just the unknot. Then show that the presentation you found is indeed a presentation of the group $\mathbf{Z}$.

Remark. The purpose of this exercise is to illustrate it's often hard to work with the fundamental group of the complement of the knot, even though we can obtain an explicit presentation for the group.

Problem 8. Using your favorite diagrams for the trefoil and figure eight, compute presentations for the fundamental groups of their complements. Can you show that they are distinct groups?

Problem 9. In this exercise, we'll relate tricolorability of a knot to the fundamental group of the complement.

(a) Let S_3 denote the third symmetric group; the group of permutations of three objects. Recall that a *transposition* is a permutation that exchanges two objects but keeps all others fixed. Let R denote the transposition (12), i.e. it exchanges the first and second objects and keeps the third fixed. Similarly let $G = (13)$ and $B = (23)$. Check that if you conjugate one of these three elements by a different one, you obtain the third one. For example: $BRB^{-1} = G$ and $R^{-1}GR = B$.

(b) Given a diagram for a knot, recall that each arc of the diagram corresponds to a generator of the Wirtinger presentation. Show that the tricoloring determines a homomorphism from the fundamental group of complement of the knot to S_3.

(c) Characterize which homomorphisms from the fundamental group of the complement of the knot to S_3 correspond to tricolorings.

Can you now solve Problem 8? Note that this exercise gives a different proof that tricolorability is an invariant of the knot.