Tutorial topic: Category Theory

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Since Eilenberg and Mac Lane introduced category theory in the 1940s, the subject has developed rapidly and become a fundamental part of modern mathematics. Category theory provides a framework to study mathematical structures and constructions, and makes precise analogies between different areas.

If you've studied set theory and linear algebra, then you are probably familiar with the disjoint union of two sets and the direct sum of two vector spaces. These constructions might seem dissimilar, but phrasing their key properties diagrammatically reveals that both are special cases of a general categorical construction. This is just one example of how a categorical perspective can unify concepts.

This tutorial will introduce the tools of category theory, focusing on examples tailored to participants' backgrounds. Fundamental concepts such as categories, functors, natural transformations, representability, limits and colimits, and adjunctions will be covered in detail. Participants will become comfortable thinking in terms of arrows, diagrams and universal properties, and reformulating mathematical statements categorically.

For the basics, we will roughly follow Mac Lane's "Categories for the Working Mathematician." We will then venture into more advanced topics determined by participants' backgrounds and interests. Additional advanced topics may include:

1. **Kan extensions.** Right and left Kan extensions subsume and generalize many constructions that we will discuss in this course, so this topic would be a natural continuation of foundational material. We would discuss generalities on Kan extensions, and make explicit the connection between this construction and others discussed in the course.
2. **Abelian categories and homological algebra.** Abelian categories are categories with a good notion of things like kernels, cokernels, and exact sequences. Homological algebra, a framework for studying (co)homology, can be formulated in such categories. This option would involve becoming comfortable with abelian categories, developing key homological algebra tools in abelian categories, and working with key examples like the category of modules over a ring.
3. **Model categories.** Model categories are categories equipped with additional structure which provide a setting to "do homotopy theory." The idea for this option would be to define model categories and their homotopy categories, and study important model categories like simplicial sets, topological spaces, and/or chain complexes.
4. **Simplicial sets and quasicategories.** Simplicial sets are combinatorial gadgets which are closely related to both topological spaces and categories. This topic would involve learning about simplicial sets and their relation to other areas of mathematics. An ambitious goal would be to understand why quasi-categories (special simplicial sets) are a good place to do "higher category theory," and study quasi-category analogues for categorical concepts studied in the course.

The formal prerequisites for studying category theory are few, but students will benefit from exposure to areas where category theory can be readily applied. For example, familiarity with abstract algebra and basic algebraic topology would be helpful.