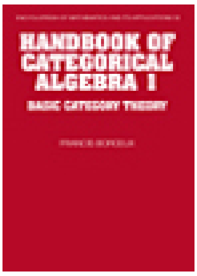

Handbook of Categorical Algebra

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1

The language of categories

1.1 Logical foundations of the theory

It is a common practice, when developing mathematics, to consider a statement involving "all groups" or "all topological spaces" .... For example we say that an abelian group A is projective when, for every surjective homomorphism of abelian groups $f: B \longrightarrow C$ and every group homomorphism $g: A \longrightarrow C$, g factors through f (see diagram 1.1). This definition of "A being projective" starts thus with a list of universal quantifiers

$$\forall B \;\forall C \;\forall f \;\forall g \;\ldots$$

This formula, from the point of view of set theory, creates a problem: the variables B and C are "running through something (= the collection of all abelian groups) which is not a set". This last fact is an easy consequence of the following well-known paradox.

Proposition 1.1.1 *There exists no set S such that*

$$x \in S \Leftrightarrow x \text{ is a set.}$$

Proof In other (bad) words: "the set of sets does not exist"! To prove this, let us assume such an S exists. Since $x \notin x$ is a formula of set theory

$$T = \{x \,|\, x \in S \;\text{ and }\; x \notin x\}$$

defines a subset T of S, thus in particular a set T. The law of excluded middle tells us that

$$T \in T \;\text{ or }\; T \notin T.$$

But from the definition of T itself we conclude that

$$T \in T \Rightarrow T \notin T,$$

$$\begin{tikzcd} & A \arrow[dl, dotted] \arrow[d, "g"] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

Diagram 1.1

$$T \notin T \Rightarrow T \in T,$$

thus in both cases a contradiction. □

Category theory will in fact be handling all the time "the collection of all groups", "the collection of all sets", "the collection of all topological spaces", and so on .... Therefore it is useful to pay some attention to these questions of "size" at the very beginning of this book.

A first way to handle, in category theory, problems of this type is to assume the axiom on the existence of "universes".

Definition 1.1.2 *A universe is a set $\mathcal{U}$ with the following properties*

(1) $x \in y$ and $y \in \mathcal{U} \;\Rightarrow\; x \in \mathcal{U}$,
(2) $I \in \mathcal{U}$ and $\forall i \in I\; x_i \in \mathcal{U} \Rightarrow \bigcup_{i \in I} x_i \in \mathcal{U}$,
(3) $x \in \mathcal{U} \Rightarrow \mathcal{P}(x) \in \mathcal{U}$,
(4) $x \in \mathcal{U}$ and $f: x \longrightarrow y$ *surjective function* $\Rightarrow y \in \mathcal{U}$,
(5) $\mathbb{N} \in \mathcal{U}$,

where $\mathbb{N}$ *denotes the set of finite ordinals and* $\mathcal{P}(x)$ *denotes the set of subsets of* x.

Notice some easy consequences of the definition.

Proposition 1.1.3

(1) $x \in \mathcal{U}$ and $y \subseteq x \Rightarrow y \in \mathcal{U}$,
(2) $x \in \mathcal{U}$ and $y \in \mathcal{U} \Rightarrow \{x, y\} \in \mathcal{U}$,
(3) $x \in \mathcal{U}$ and $y \in \mathcal{U} \Rightarrow x \times y \in \mathcal{U}$,
(4) $x \in \mathcal{U}$ and $y \in \mathcal{U} \Rightarrow x^y \in \mathcal{U}$.

Proof We prove (1) and leave the rest as an easy exercise. First of all $\emptyset \in \mathbb{N}$ and $\mathbb{N} \in \mathcal{U}$, thus $\emptyset \in \mathcal{U}$. Now if $x \in \mathcal{U}$ and $y \subseteq x$ with $y \neq \emptyset$, choose $z \in y$. Define $f: x \longrightarrow y$ to be

$$f(t) = t \text{ if } t \in y,$$
$$f(t) = z \text{ if } t \notin y.$$

Obviously f is surjective and therefore $y \in \mathcal{U}$. $\square$

It should be noticed that – assuming the axiom of choice in our set theory – condition (4) in definition 1.1.2 could have been replaced precisely by

$$x \in \mathcal{U} \text{ and } y \subseteq x \Rightarrow y \in \mathcal{U}.$$

Now the axiom on the existence of universes is just

Axiom 1.1.4 *Every set belongs to some universe.*

Not much is known about this axiom from the point of view of set theory. Because of the property

$$x \in \mathcal{U} \text{ and } y \subseteq x \Rightarrow y \in \mathcal{U},$$

it sounds reasonable to think of the elements of a universe as being "sufficiently small sets". If you choose to use the theory of universes as a foundation for category theory, the following convention has to remain valid throughout this book.

Convention 1.1.5 *We fix a universe $\mathcal{U}$ and call "small sets" the elements of $\mathcal{U}$.*

Obviously we now have the following

Proposition 1.1.6 *There exists a set S with the property $x \in S \Leftrightarrow x$ is a small set.*

Proof Just choose $S = \mathcal{U}$. $\square$

An analogous statement is valid for small abelian groups, small topological spaces, and so on .... For example a small group is a pair $(G, +)$ where G is a small set (and there is just a set of them) and $+$ is a suitable mapping $G \times G \longrightarrow G$ (and there is just a set of them); so we can draw the conclusion by proposition 1.1.3.

An alternative way to handle these problems of size is to use the Gödel–Bernays theory of sets and classes. In the Zermelo–Fränkel theory, the primitive notions are "set" and "membership relation". In the Gödel–Bernays theory, there is one more primitive notion called "class" (think of it as "a big set"); that primitive notion is related to the other two by the property that every set is a class and, more precisely:

Axiom 1.1.7 *A class is a set if and only if it belongs to some (other) class.*

The axioms concerning classes imply in particular the following "comprehension scheme" for constructing classes.

Comprehension scheme 1.1.8 *If $\varphi(x_1, \dots, x_n)$ is a formula where quantification just occurs on set variables, there exists a class A such that*

$$(x_1, \dots, x_n) \in A \text{ if and only if } \varphi(x_1, \dots, x_n).$$

For example, there exists a class A with the property

$$(G, +) \in A \text{ if and only if } (G, +) \text{ “is a group”}$$

(where “is a group” is an abbreviation for the group axioms); in other words, this defines the “class of all groups”. In the same way we deduce the existence of the class of sets, the class of topological spaces, the class of projective abelian groups, and so on.

When the axiom of universes is assumed and a universe $\mathcal{U}$ is fixed, one gets a model of the Gödel–Bernays theory by choosing as “sets” the elements of $\mathcal{U}$ and as “classes” the subsets of $\mathcal{U}$. It makes no relevant difference whether we base category theory on the axiom of universes or on the Gödel-Bernays theory of classes. We shall use the terminology of the latter, thus using the words “set” and “class”; the reader who prefers the terminology of the former should thus read “small set” when we write “set” and should read “set” when we write “class”.

1.2 Categories and functors

With every mathematical structure on a set is generally associated a notion of “mapping compatible with that structure”: a group homomorphism between groups, a linear mapping between vector spaces, a continuous mapping between topological spaces, and so on .... The basic examples of a category are designed in precisely that way: those sets provided with a prescribed structure and, between them, those mappings which are compatible with the given strucure.

Definition 1.2.1 *A category $\mathscr{C}$ consists of the following:*

(1) a class $|\mathscr{C}|$, whose elements will be called “objects of the category”;

(2) for every pair A, B of objects, a set $\mathscr{C}(A, B)$, whose elements will be called “morphisms” or “arrows” from A to B;

(3) for every triple A, B, C of objects, a composition law

$$\mathscr{C}(A, B) \times \mathscr{C}(B, C) \longrightarrow \mathscr{C}(A, C);$$

the composite of the pair (f, g) will be written $g \circ f$ or just gf;

(4) for every object A, a morphism $1_A \in \mathscr{C}(A, A)$, called the identity on A.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

Diagram 1.2

These data are subject to the following axioms.

(1) Associativity axiom: *given morphisms* $f \in \mathscr{C}(A,B)$, $g \in \mathscr{C}(B,C)$, $h \in \mathscr{C}(C,D)$ *the following equality holds:*
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(2) Identity axiom: *given morphisms* $f \in \mathscr{C}(A,B)$, $g \in \mathscr{C}(B,C)$ *the following equalities hold:*
$$1_B \circ f = f, \quad g \circ 1_B = g.$$

A morphism $f \in \mathscr{C}(A,B)$ will often be represented by the notation $f: A \longrightarrow B$; A is called the "domain" or the "source" of f and B is called the "codomain" or the "target" of f. In the situation of diagram 1.2, we say that the given square is "commutative" when the equality $g \circ f = k \circ h$ holds between the two possible composites; an analogous terminology holds for diagrams of arbitrary shape.

As usual 1_A is the only morphism from A to A which plays the role of an identity for the composition law. Indeed if $i_A \in \mathscr{C}(A,A)$ is another such morphism
$$1_A = 1_A \circ i_A = i_A.$$

Let us now define a "homomorphism of categories".

Definition 1.2.2 *A functor* F *from a category* $\mathscr{A}$ *to a category* $\mathscr{B}$ *consists of the following:*

(1) a mapping
$$|\mathscr{A}| \longrightarrow |\mathscr{B}|$$
between the classes of objects of $\mathscr{A}$ *and* $\mathscr{B}$*; the image of* $A \in \mathscr{A}$ *is written* $F(A)$ *or just* FA*;*

(2) for every pair of objects A, A' *of* $\mathscr{A}$*, a mapping*
$$\mathscr{A}(A,A') \longrightarrow \mathscr{B}(FA,FA');$$
the image of $f \in \mathscr{A}(A,A')$ *is written* $F(f)$ *or just* Ff*.*

These data are subject to the following axioms:

(1) for every pair of morphisms $f \in \mathscr{A}(A, A')$, $g \in \mathscr{A}(A', A'')$

$$F(g \circ f) = F(g) \circ F(f);$$

(2) for every object $A \in \mathscr{A}$

$$F(1_A) = 1_{FA}.$$

Given two functors $F: \mathscr{A} \longrightarrow \mathscr{B}$ and $G: \mathscr{B} \longrightarrow \mathscr{C}$, a pointwise composition immediately produces a new functor $G \circ F: \mathscr{A} \longrightarrow \mathscr{C}$. This composition law is obviously associative. The identity functor on the category $\mathscr{A}$ (i.e. choose every mapping in definition 1.2.2 to be the identity) is clearly an identity for that composition law. A careless argument could thus lead to the conclusion that categories and functors constitute a new category ... but this can easily be reduced to a contradiction using proposition 1.1.1! The point is that, in the axioms for a category, it is required to have a *set* of morphisms between any two objects. And when the categories $\mathscr{A}$ and $\mathscr{B}$ merely have a class of objects, there is no way to force the functors from $\mathscr{A}$ to $\mathscr{B}$ to constitute a *set*. All along in this book we shall realize how crucial it is, in category theory, to distinguish all the time between sets and classes. To facilitate the language, we particularize definition 1.2.1.

Definition 1.2.3 *A category $\mathscr{C}$ is called a small category when its class $|\mathscr{C}|$ of objects is a set.*

The next result is then obvious (see 1.1.8).

Proposition 1.2.4 *Small categories and functors between them constitute a category.* □

Examples 1.2.5

Let us first list some obvious examples of categories and the corresponding notation, when it is classical.

1.2.5.a Sets and mappings: Set.

1.2.5.b Topological spaces and continuous mappings: Top.

1.2.5.c Groups and group homomorphisms: Gr.

1.2.5.d Commutative rings with unit and ring homomorphisms: Rng.

1.2.5.e Real vector spaces and linear mappings: $\mathsf{Vect}_{\mathbb{R}}$.

1.2.5.f Real Banach spaces and bounded linear mappings: Ban_{∞}.

1.2.5.g Sets and injective mappings.

1.2.5.h Real Banach spaces and linear contractions: Ban_1.

And so on.

Examples 1.2.6

Here is a list of some mathematical devices which can also be seen as categories.

1.2.6.a Choose as objects the natural numbers and as arrows from n to m the matrices with n rows and m columns; the composition is the usual product of matrices.

1.2.6.b A poset $(X, \leq)$ can be viewed as a category $\mathscr{X}$ whose objects are the elements of X; the set $\mathscr{X}(x, y)$ of morphisms is a singleton when $x \leq y$ and is empty otherwise. The possibility of defining a (unique) composition law is just the transitivity axiom of the partial order; the existence of identities is just the reflexivity axiom.

1.2.6.c Every set X can be viewed as a category $\mathscr{X}$ whose objects are the elements of X and the only morphisms are identities. ($\mathscr{X}(x, y)$ is a singleton when $x = y$ and is empty otherwise). A category whose only morphisms are the identities is called a discrete category.

1.2.6.d A monoid $(M, \cdot)$ can be seen as a category $\mathscr{M}$ with a single object $*$ and $M = \mathscr{M}(*, *)$ as a set of morphisms; the composition law is just the multiplication of the monoid. As a special case, we can view any group as a category. When a ring with unit is considered as a special case of a category, the composition law of that category is generally that induced by the multiplication of the ring.

Examples 1.2.7

From a given category $\mathscr{C}$, one very often constructs new categories of "diagrams in $\mathscr{C}$". Here are some basic contructions.

1.2.7.a Let us fix an object $I \in \mathscr{C}$. The category $\mathscr{C}/I$ of "arrows over I" is defined by the following.

- Objects: the arrows of $\mathscr{C}$ with codomain I.
- Morphisms from the object $(f: A \longrightarrow I)$ to the object $(g: B \longrightarrow I)$: the morphisms $h: A \longrightarrow B$ in $\mathscr{C}$ such that $g \circ h = f$ (the "commutative triangles over I"); see diagram 1.3.

The composition law is that induced by the composition of $\mathscr{C}$. Notice that when $\mathscr{C}$ is the category of sets and mappings, a mapping $f: A \longrightarrow I$ can be identified with the I-indexed family of sets $\left(f^{-1}(i)\right)_{i \in I}$ so that the previous category is just that of I-indexed families of sets and I-indexed families of mappings.

1.2.7.b Again fixing an object $I \in \mathscr{C}$, we define the category $I/\mathscr{C}$ of "arrows under I".

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

Diagram 1.3

$$\begin{tikzcd} & I \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "h"'] & & B \end{tikzcd}$$

Diagram 1.4

- Objects: the arrows of $\mathscr{C}$ with domain I.
- Morphisms from the object $f\colon I \longrightarrow A$ to the object $g\colon I \longrightarrow B$: the morphisms $h\colon A \longrightarrow B$ in $\mathscr{C}$ such that $h \circ f = g$ (the "commutative triangles under I"); see diagram 1.4.

The composition law is induced by that of $\mathscr{C}$.

1.2.7.c The category $\mathsf{Ar}(\mathscr{C})$ of arrows of $\mathscr{C}$ has for objects all the arrows of $\mathscr{C}$; a morphism from the object $(f\colon A \longrightarrow B)$ to the object $(g\colon C \longrightarrow D)$ is a pair $(h\colon A \longrightarrow C, k\colon B \longrightarrow D)$ of morphisms of $\mathscr{C}$, with the property $k \circ f = g \circ h$ ("a commutative square"); see diagram 1.5. Again, the composition law is that induced pointwise by the composition in $\mathscr{C}$.

In examples 1.2.7.a,b,c, it is easy to check that when $\mathscr{C}$ is small, so are the three categories $\mathscr{C}/I$, $I/\mathscr{C}$, $\mathsf{Ar}(\mathscr{C})$.

Examples 1.2.8

Let us finally mention some first examples of functors.

1.2.8.a The "forgetful functor" $U\colon \mathsf{Ab} \longrightarrow \mathsf{Set}$ from the category Ab of abelian groups to the category Set of sets maps a group $(G, +)$ to the underlying set G and a group homomorphism f to the corresponding mapping f.

1.2.8.b If R is a commutative ring, let us write Mod_R for the category of R-modules and R-linear mappings. Tensoring with R produces a functor from the category Ab of abelian groups to Mod_R:

$$- \otimes R\colon \mathsf{Ab} \longrightarrow \mathsf{Mod}_R.$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Diagram 1.5

An abelian group A is mapped to the group $A \otimes_{\mathbb{Z}} R$ provided with the scalar multiplication induced by the formula

$$(a \otimes r)r' = a \otimes (rr').$$

A group homomorphism $f: A \longrightarrow B$ is mapped to the R-linear mapping $f \otimes \mathrm{id}_R$.

1.2.8.c We obtain a functor $\mathcal{P}: \mathsf{Set} \longrightarrow \mathsf{Set}$ from the category of sets to itself by mapping a set A to its power set $\mathcal{P}(A)$ and a mapping $f: A \longrightarrow B$ to the "direct image mapping" from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.

1.2.8.d Given a category $\mathscr{C}$ and a fixed object $C \in \mathscr{C}$, we define a functor

$$\mathscr{C}(C, -): \mathscr{C} \longrightarrow \mathsf{Set}$$

from $\mathscr{C}$ to the category of sets by first putting

$$\mathscr{C}(C, -)(A) = \mathscr{C}(C, A).$$

Now if $f: A \longrightarrow B$ is a morphism of $\mathscr{C}$, the corresponding mapping

$$\mathscr{C}(C, -)(f) \equiv \mathscr{C}(C, f): \mathscr{C}(C, A) \longrightarrow \mathscr{C}(C, B)$$

is defined by the formula

$$\mathscr{C}(C, f)(g) = f \circ g$$

for an arrow $g \in \mathscr{C}(C, A)$. Such a functor is called a "representable functor" (the functor is "represented" by the object C).

1.2.8.e Given two categories $\mathscr{A}$, $\mathscr{B}$ and a fixed object $B \in \mathscr{B}$, we define the "constant functor to B"

$$\Delta_B: \mathscr{A} \longrightarrow \mathscr{B}$$

by

$$\Delta_B(A) = B, \ \ \Delta_B(f) = 1_B$$

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FA' \arrow[r, "\alpha_{A'}"'] & GA' \end{tikzcd}$$

Diagram 1.6

for every object $A \in \mathscr{A}$ and every morphism f of $\mathscr{A}$.

1.3 Natural transformations

General topology studies, in particular, topological spaces and continuous functions between them. But given two continuous functions from a space to another one, there exists also the notion of a "homotopy" between those two continuous functions, which allows you to "pass" from one function to the other one. A similar situation exists for categories and functors.

Definition 1.3.1 *Consider two functors $F, G: \mathscr{A} \rightrightarrows \mathscr{B}$ from a category $\mathscr{A}$ to a category $\mathscr{B}$. A natural transformation $\alpha: F \Rightarrow G$ from F to G is a class of morphisms $(\alpha_A: FA \longrightarrow GA)_{A \in \mathscr{A}}$ of $\mathscr{B}$ indexed by the objects of $\mathscr{A}$ and such that for every morphism $f: A \longrightarrow A'$ in $\mathscr{A}$, $\alpha_{A'} \circ F(f) = G(f) \circ \alpha_A$. (see diagram 1.6)*

It is an obvious matter to notice that, when F, G, H are functors from $\mathscr{A}$ to $\mathscr{B}$ and $\alpha: F \Rightarrow G$, $\beta: G \Rightarrow H$ are natural tranformations, the formula

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$$

defines a new natural transformation $\beta \circ \alpha: F \Rightarrow H$. That composition law is clearly associative and possesses a unit at each functor F: this is just the natural transformation 1_F whose A-component is 1_{FA}. Again a careless argument would deduce the existence of a category whose objects are the functors from $\mathscr{A}$ to $\mathscr{B}$ and whose morphisms are the natural transformations between them. But since $\mathscr{A}$ and $\mathscr{B}$ have merely classes of objects, there is in general no way to prove the existence of a set of natural transformations between two functors! But when $\mathscr{A}$ is small, that problem disappears and we get the following result.

Proposition 1.3.2 *Let $\mathscr{A}$ be a small category and $\mathscr{B}$ an arbitrary category. The functors from $\mathscr{A}$ to $\mathscr{B}$ and the natural transformations between them constitute a category; that category is small as long as $\mathscr{B}$ is small.* □

We prove now the first important theorem of this book. We refer to example 1.2.8.d for the description of the representable functors.

Theorem 1.3.3 (The Yoneda lemma)
Consider a functor $F: \mathscr{A} \longrightarrow \mathsf{Set}$ from an arbitrary category $\mathscr{A}$ to the category of sets and mappings, an object $A \in \mathscr{A}$ and the corresponding representable functor $\mathscr{A}(A,-): \mathscr{A} \longrightarrow \mathsf{Set}$. There exists a bijective correspondence

$$\theta_{F,A}: \mathsf{Nat}\big(\mathscr{A}(A,-), F\big) \xrightarrow{\cong} FA$$

between the natural transformations from $\mathscr{A}(A,-)$ to F and the elements of the set FA; in particular those natural transformations constitute a set. The bijections $\theta_{F,A}$ constitute a natural transformation in the variable A; when $\mathscr{A}$ is a small category, the bijections $\theta_{F,A}$ also constitute a natural transformation in the variable F.

Proof For a given natural transformation $\alpha: \mathscr{A}(A,-) \Rightarrow F$, we define $\theta_{F,A}(\alpha) = \alpha_A(1_A)$. With a given element $a \in FA$ we associate, for every object $B \in \mathscr{A}$, a mapping

$$\tau(a)_B: \mathscr{A}(A,B) \longrightarrow FB$$

defined by $\tau(a)_B(f) = F(f)(a)$. This class of mappings defines a natural transformation

$$\tau(a): \mathscr{A}(A,-) \Rightarrow F$$

since, for every morphism $g: B \longrightarrow C$ in $\mathscr{A}$, the relation

$$Fg \circ \tau(a)_B = \tau(a)_C \circ \mathscr{A}(A,g)$$

(see diagram 1.7) reduces to the equality

$$\forall f \in \mathscr{A}(A,B) \;\; F(g \circ f)(a) = Fg\big((Ff)(a)\big),$$

which follows from the functoriality of F.

$\theta_{F,A}$ and τ are inverse to each other. Indeed, starting from $a \in FA$ we have

$$\theta_{F,A}\big(\tau(a)\big) = \tau(a)_A(1_A) = F(1_A)(a) = 1_{FA}(a) = a.$$

$$\begin{array}{ccc} \mathscr{A}(A,B) & \xrightarrow{\tau(a)_B} & FB \\ {\scriptstyle \mathscr{A}(A,g)}\downarrow & & \downarrow{\scriptstyle Fg} \\ \mathscr{A}(A,C) & \xrightarrow[\tau(a)_C]{} & FC \end{array}$$

Diagram 1.7

On the other hand, starting from $\alpha: \mathscr{A}(A,-) \Rightarrow F$ and choosing a morphism $f: A \longrightarrow B$ in $\mathscr{A}$,

$$\begin{aligned} \tau\big(\theta_{F,A}(\alpha)\big)_B(f) &= \tau\big(\alpha_A(1_A)\big)_B(f) \\ &= F(f)\big(\alpha_A(1_A)\big) \\ &= \alpha_B\big(\mathscr{A}(A,f)(1_A)\big) \\ &= \alpha_B(f \circ 1_A) \\ &= \alpha_B(f), \end{aligned}$$

where the third equality follows from the naturality of α. This proves the first part of the theorem.

To prove the naturality of the bijections, let us consider the functor $N: \mathscr{A} \longrightarrow \mathsf{Set}$ defined by

$$N(A) = \mathsf{Nat}\big(\mathscr{A}(A,-), F\big).$$

and for every morphism $f: A \longrightarrow B$ in $\mathscr{A}$

$$\begin{aligned} &N(f): \mathsf{Nat}\big(\mathscr{A}(A,-), F\big) \longrightarrow \mathsf{Nat}\big(\mathscr{A}(B,-), F\big) \\ &N(f)(\alpha) = \alpha \circ \mathscr{A}(f,-) \end{aligned}$$

(see example 1.3.6.c for the definition of $\mathscr{A}(f,-)$). We are claiming the existence of a natural transformation $\eta: N \Rightarrow F$ defined by $\eta_A = \theta_{F,A}$. Indeed, with the previous notation,

$$\begin{aligned} \big(\theta_{F,B} \circ N(f)\big)(\alpha) &= \theta_{F,B}\big(\alpha \circ \mathscr{A}(f,-)\big) \\ &= \big(\alpha \circ \mathscr{A}(f,-)\big)_B(1_B) \\ &= \alpha_B(f), \\ \big(F(f) \circ \theta_{F,A}\big)(\alpha) &= F(f)\big(\alpha_A(1_A)\big) \\ &= \big(\alpha_B \circ \mathscr{A}(A,f)\big)(1_A) \\ &= \alpha_B(f). \end{aligned}$$

Moreover, when $\mathscr{A}$ is a small category, it makes sense to consider the category $\mathsf{Fun}(\mathscr{A}, \mathsf{Set})$ of functors from $\mathscr{A}$ to Set and natural transformations between them. For a fixed object $A \in \mathscr{A}$ we consider this time the functor $M\colon \mathsf{Fun}(\mathscr{A}, \mathsf{Set}) \longrightarrow \mathsf{Set}$ defined by

$$M(F) = \mathsf{Nat}\big(\mathscr{A}(A,-), F\big);$$

for a functor $G\colon \mathscr{A} \longrightarrow \mathsf{Set}$ and a natural transformation $\gamma\colon F \Rightarrow G$,

$$M(\gamma)\colon \mathsf{Nat}\big(\mathscr{A}(A,-), F\big) \longrightarrow \mathsf{Nat}\big(\mathscr{A}(A,-), G\big)$$

is defined by $M(\gamma)(\alpha) = \gamma \circ \alpha$. On the other hand we consider the functor "evaluation in A" $\mathrm{ev}_A\colon \mathsf{Fun}(\mathscr{A}, \mathsf{Set}) \longrightarrow \mathsf{Set}$ defined by

$$\mathrm{ev}_A(F) = FA, \quad \mathrm{ev}_A(\gamma) = \gamma_A.$$

We claim to have a natural transformation $\mu\colon M \Rightarrow \mathrm{ev}_A$ defined by $\mu_F = \theta_{F,A}$. Indeed, with the previous notation,

$$\begin{aligned}\big(\theta_{G,A} \circ M(\gamma)\big)(\alpha) &= \theta_{G,A}(\gamma \circ \alpha)\\ &= (\gamma \circ \alpha)_A(1_A),\\ \big(\mathrm{ev}_A(\gamma) \circ \theta_{F,A}\big)(\alpha) &= \gamma_A\big(\alpha_A(1_A)\big).\end{aligned}$$ □

In proposition 1.3.2 we have used a first composition law for natural transformations. In fact, there exists another possible type of composition for natural transformations.

Proposition 1.3.4 *Consider the following situation:*

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{C} \end{tikzcd}$$

where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are categories, F, G, H, K are functors and α, β are natural transformations. The formula, for every $A \in \mathscr{A}$,

$$(\beta * \alpha)_A = \beta_{GA} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{FA}$$

defines a natural transformation

$$\beta * \alpha\colon H \circ F \Rightarrow K \circ G.$$

called the "Godement product" of the two natural transformations α and β.

Proof $(\beta * \alpha)_A$ is thus defined considering diagram 1.8 which is indeed commutative by naturality of β. The proposition asserts, for every morphism $f\colon A \longrightarrow A'$ in $\mathscr{A}$, the commutativity of the outer rectangle

$$\begin{tikzcd} HFA \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{FA}"'] & HGA \arrow[d, "\beta_{GA}"] \\ KFA \arrow[r, "K(\alpha_A)"'] & KGA \end{tikzcd}$$

Diagram 1.8

$$\begin{tikzcd} HFA \arrow[r, "H(\alpha_A)"] \arrow[d, "HFf"'] & HGA \arrow[r, "\beta_{GA}"] \arrow[d, "HGf"'] & KGA \arrow[d, "KGf"] \\ HFA' \arrow[r, "H(\alpha_{A'})"'] & HGA' \arrow[r, "\beta_{GA'}"'] & KGA' \end{tikzcd}$$

Diagram 1.9

in diagram 1.9. It holds since the first square commutes by naturality of α and functoriality of H and the second square commutes by naturality of β. □

The proof of the next proposition is a straightforward exercise left to the reader.

Proposition 1.3.5 *Consider the situation*

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"] \arrow[r, "H"] \arrow[r, bend right=50, "L"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma"] & \mathscr{B} \arrow[r, bend left=50, "G"] \arrow[r, "K"] \arrow[r, bend right=50, "M"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathscr{C} \end{tikzcd}$$

where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are categories, F, G, H, K, L, M are functors and α, β, γ, δ are natural transformations. The following equality holds:

$$(\delta * \gamma) \circ (\beta * \alpha) = (\delta \circ \beta) * (\gamma \circ \alpha).$$ □

For the sake of brevity and with the notations of the previous propositions, we shall often write $\beta * F$ instead of $\beta * 1_F$ or $G * \alpha$ instead of $1_G * \alpha$.

Examples 1.3.6

1.3.6.a Consider the power set functor $\mathcal{P}\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ defined in 1.2.8.c and the identity functor $\mathsf{id}\colon \mathsf{Set} \longrightarrow \mathsf{Set}$. The mappings "singleton"

$$\sigma_E\colon E \longrightarrow \mathcal{P}(E)$$

which map an element $x \in E$ to the singleton $\{x\}$ constitute a natural transformation $\sigma\colon \mathsf{id} \Rightarrow \mathcal{P}$.

1.3.6.b Consider the category $\mathsf{Vect}_{\mathbb{R}}$ of real vector spaces and the bidual functor

$$(\ )^{**}\colon \mathsf{Vect}_{\mathbb{R}} \longrightarrow \mathsf{Vect}_{\mathbb{R}}.$$

The canonical morphisms

$$\sigma_V\colon V \longrightarrow V^{**},\quad v \mapsto v^{**}$$

for every vector space V, define a natural transformation from the identity functor to the bidual functor.

1.3.6.c Consider a category $\mathscr{A}$ and a morphism $f\colon A \longrightarrow B$ of $\mathscr{A}$. We obtain a natural transformation

$$\mathscr{A}(f,-)\colon \mathscr{A}(B,-) \Rightarrow \mathscr{A}(A,-)$$

between the functors represented by A and B (see 1.2.8.d) by putting, for every object $C \in \mathscr{A}$ and every morphism $g \in \mathscr{A}(B,C)$,

$$\mathscr{A}(f,-)_C(g) = g \circ f.$$

Generally we shall write $\mathscr{A}(f,C)$ for the mapping $\mathscr{A}(f,-)_C$.

1.3.6.d Given two categories $\mathscr{A}$, $\mathscr{B}$ and a fixed morphism $b\colon B \longrightarrow B'$, we define the "constant natural transformation on b" $\Delta_b\colon \Delta_B \Rightarrow \Delta_{B'}$ by $(\Delta_b)_A = b$ for every object $A \in \mathscr{A}$ (see 1.2.8.e for the definition of Δ_B, $\Delta_{B'}$).

1.4 Contravariant functors

If $\mathscr{A}$ is a small category, we know it makes sense to consider the category of functors from $\mathscr{A}$ to Set and natural transformations between them (see 1.3.2). In examples 1.2.8.d and 1.3.6.c we have defined a mapping

$$\begin{aligned} &Y^*\colon \mathscr{A} \longrightarrow \mathsf{Fun}(\mathscr{A},\mathsf{Set}),\\ &Y^*(A) = \mathscr{A}(A,-),\ \ Y^*(f) = \mathscr{A}(f,-), \end{aligned}$$

where $A \in |\mathscr{A}|$ is an object of $\mathscr{A}$ and f is a morphism of $\mathscr{A}$. It is rather obvious that, given morphisms

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

in $\mathscr{A}$, we obtain the following equalities:

$$Y^*(g \circ f) = Y^*(f) \circ Y^*(g),\ \ Y^*(1_B) = 1_{Y^*B}.$$

So Y^* is a mapping which "reverses the direction of every morphism",

$$f: A \longrightarrow B, \quad Y^*(f): Y^*(B) \longrightarrow Y^*(A),$$

and – up to this reversing process – preserves the composition law and identities. This is what we shall call a "contravariant functor".

Definition 1.4.1 *A contravariant functor F from a category $\mathscr{A}$ to a category $\mathscr{B}$ consists of the following:*

(1) a mapping

$$|\mathscr{A}| \longrightarrow |\mathscr{B}|$$

between the classes of objects; the image of $A \in \mathscr{A}$ is written $F(A)$ or just FA;

(2) for every pair of objects $A, A' \in \mathscr{A}$, a mapping

$$\mathscr{A}(A, A') \longrightarrow \mathscr{B}(FA', FA);$$

the image of $f \in \mathscr{A}(A, A')$ is written $F(f)$ or just Ff.

These data are subject to the following axioms:

(1) for every pair of morphisms $f \in \mathscr{A}(A, A')$, $g \in \mathscr{A}(A', A'')$,

$$F(g \circ f) = F(f) \circ F(g);$$

(2) for every object $A \in \mathscr{A}$,

$$F(1_A) = 1_{FA}.$$

When confusion could be possible, we shall emphasize the fact that we are definitely working with a functor in the sense of definition 1.2.2 by calling it a *covariant* functor.

The notion of a natural transformation can easily be carried over to the contravariant case.

Definition 1.4.2 *Consider two contravariant functors $F, G: \mathscr{A} \rightrightarrows \mathscr{B}$ from a category $\mathscr{A}$ to a category $\mathscr{B}$. A natural transformation $\alpha: F \Rightarrow G$ from F to G is a class of morphisms $(\alpha_A: FA \longrightarrow GA)_{A \in \mathscr{A}}$ of $\mathscr{B}$ indexed by the objects of $\mathscr{A}$ and such that for every morphism $f: A \longrightarrow A'$ in $\mathscr{A}$, $G(f) \circ \alpha_{A'} = \alpha_A \circ F(f)$ (see diagram 1.10).*

All the results of sections 1.2 and 1.3 can be transposed to the contravariant case; this is a straightforward exercise left to the reader. Moreover, we should mention at this point that the validity of this transposition can also be obtained as an application of the duality principle of section 1.10.

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] & GA \\ FA' \arrow[u, "Ff"] \arrow[r, "\alpha_{A'}"'] & GA' \arrow[u, "Gf"'] \end{tikzcd}$$

Diagram 1.10

Examples 1.4.3

1.4.3.a We started this section with the example of the "contravariant Yoneda embedding"

$$Y^*\colon \mathscr{A} \longrightarrow \mathsf{Fun}(\mathscr{A}, \mathsf{Set})$$

for a small category $\mathscr{A}$.

1.4.3.b Example 1.2.8.d can be "dualized"; given a category $\mathscr{A}$ and an object $A \in \mathscr{A}$ we define a contravariant functor

$$\mathscr{A}(-, A)\colon \mathscr{A} \longrightarrow \mathsf{Set}$$

by the formulas

$$\mathscr{A}(-, A)(B) = \mathscr{A}(B, A)$$

for every object $B \in \mathscr{A}$, and

$$\mathscr{A}(-, A)(f)\colon \mathscr{A}(C, A) \longrightarrow \mathscr{A}(B, A),$$
$$\mathscr{A}(-, A)(f)(g) = g \circ f$$

for all morphisms $f\colon B \longrightarrow C$ and $g\colon C \longrightarrow A$ in $\mathscr{A}$.

1.4.3.c Example 1.3.6.c can be "dualized" as well. With the previous notation we obtain a natural transformation

$$\mathscr{A}(-, f)\colon \mathscr{A}(-, B) \longrightarrow \mathscr{A}(-, C)$$

for $f\colon B \longrightarrow C$, by putting

$$\mathscr{A}(-, f)_D(h) = f \circ h$$

for every object D and every morphism $h\colon D \longrightarrow B$. Generally, we shall write $\mathscr{A}(D, f)$ for the mapping $\mathscr{A}(-, f)_D$.

1.4.3.d Again using the previous notation, example 1.4.3.a itself can be "dualized". Let us write $\mathsf{Fun}^*(\mathscr{A}, \mathsf{Set})$ for the category of contravariant functors from a small category $\mathscr{A}$ to Set. The "covariant Yoneda embedding" is the covariant functor

$$Y_*\colon \mathscr{A} \longrightarrow \mathsf{Fun}^*(\mathscr{A}, \mathsf{Set})$$

defined by the formulas

$$Y_*(A) = \mathscr{A}(-, A),$$
$$Y_*(f) = \mathscr{A}(-, f)$$

for every object $A \in \mathscr{A}$ and every morphism f of $\mathscr{A}$.

1.4.3.e Consider the category Rng of commutative rings with unit and the category Top of topological spaces and continuous mappings. The construction of the Zariski spectrum of a ring gives rise to a contravariant functor

$$\mathsf{Sp}\colon \mathsf{Rng} \longrightarrow \mathsf{Top}.$$

For a given ring A, $\mathsf{Sp}(A)$ is the Zariski spectrum of A, that is the set of prime ideals of A provided with the topology generated by the fundamental open subsets

$$\mathcal{O}_a = \{P \in \mathsf{Sp}(A) \,|\, a \notin P\}$$

for every element $a \in A$. For a given ring homomorphism $f\colon A \longrightarrow B$, the inverse image process maps a prime ideal of B to a prime ideal of A; therefore we get a mapping

$$\mathsf{Sp}(f)\colon \mathsf{Sp}(B) \longrightarrow \mathsf{Sp}(A),$$
$$\mathsf{Sp}(f)(P) = f^{-1}(P),$$

which is easily proved to be continuous.

1.4.3.f The last example in this section is that of a contravariant functor $\mathcal{P}^*\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ which coincides on the objects with the covariant functor $\mathcal{P}\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ defined in 1.2.8.c. Thus $\mathcal{P}^*(X)$ is the power set of X and for a given mapping $f\colon X \longrightarrow Y$,

$$\mathcal{P}^*(f)\colon \mathcal{P}^*(Y) \longrightarrow \mathcal{P}^*(X), \quad \mathcal{P}^*(f)(U) = f^{-1}(U)$$

is the inverse image mapping.

1.5 Full and faithful functors

An abelian group is a set provided with some additional structure; a group homomorphism is a mapping which satisfies some additional property. So, in some vague sense, the category of abelian groups is "included" in the category of sets... the expected "inclusion" being the functor described in example 1.2.8.a. But this functor is by no means injective since on the same set G, there exist in general many different abelian group structures. In fact this functor is what we shall call a "faithful functor".

Definition 1.5.1 *Consider a functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ and for every pair of objects A, $A' \in \mathscr{A}$, the mapping*

$$\mathscr{A}(A,A') \longrightarrow \mathscr{B}(FA,FA'),\;\; f \mapsto Ff.$$

(1) The functor F is faithful when the abovementioned mappings are injective for all A, A'.

(2) The functor F is full when the abovementioned mappings are surjective for all A, A'.

(3) The functor F is full and faithful when the abovementioned mappings are bijective for all A, A'.

(4) The functor F is an isomorphism of categories when it is full and faithful and induces a bijection $|\mathscr{A}| \longrightarrow |\mathscr{B}|$ on the classes of objects.

The reader will easily adapt definition 1.5.1 to the case of contravariant functors. Definiton 1.5.1.4 is a special instance, in the category of small categories and functors, of the general notion of isomorphism in a category.

Proposition 1.5.2 *The Yoneda embedding functors described in examples 1.4.3.a,d are full and faithful functors.*

Proof In the case of the contravariant Yoneda embedding, we have to prove that given two objects A, B in a small category A, the canonical mapping

$$\mathscr{A}(A,B) \longrightarrow \mathsf{Nat}\big(\mathscr{A}(B,-),\mathscr{A}(A,-)\big),\;\; f \mapsto \mathscr{A}(f,-)$$

is bijective. This is a special case of the Yoneda lemma (see 1.3.3) applied to the functor $\mathscr{A}(A,-)$ and the object B.

The case of the covariant embedding is proved in a "dual" way. □

Let us conclude with some terminology concerning subcategories.

Definition 1.5.3 *A subcategory $\mathscr{B}$ of a category $\mathscr{A}$ consists of:*

(1) a subclass $|\mathscr{B}| \subseteq |\mathscr{A}|$ of the class of objects,

(2) for every pair of objects $A, A' \in \mathscr{A}$, a subset $\mathscr{B}(A,A') \subseteq \mathscr{A}(A,A')$,

in such a way that

(1) $f \in \mathscr{B}(A,A')$ and $g \in \mathscr{B}(A',A'') \Rightarrow g \circ f \in \mathscr{B}(A,A'')$,

$$\begin{tikzcd}
FA \arrow[r, "f"] \arrow[d, "Fa"'] & GB \arrow[d, "Gb"] \\
FA' \arrow[r, "f'"'] & GB'
\end{tikzcd}$$

Diagram 1.11

(2) $\forall A \in \mathscr{B}$, $1_A \in \mathscr{B}(A, A)$.

A subcategory $\mathscr{B}$ of $\mathscr{A}$ thus gives rise to an injective (and therefore faithful) inclusion functor $\mathscr{B} \longrightarrow \mathscr{A}$.

Definition 1.5.4 *A subcategory $\mathscr{B}$ of a category $\mathscr{A}$ is called a full subcategory when the inclusion functor $\mathscr{B} \longrightarrow \mathscr{A}$ is also a full functor.*

$\mathscr{B}$ is thus full in $\mathscr{A}$ when

$$A, A' \in \mathscr{B} \Rightarrow \mathscr{B}(A, A') = \mathscr{A}(A, A').$$

The category of sets and injections between them is a (non-full) subcategory of the category of sets and mappings. The category of finite sets and mappings between them is a full subcategory of the category of sets and mappings. A full subcategory can clearly be defined by just giving its class of objects.

1.6 Comma categories

We indicate now a quite general process for constructing new categories from given ones. This type of construction will be used very often in this book.

Definition 1.6.1 *Consider two functors $F: \mathscr{A} \longrightarrow \mathscr{C}$ and $G: \mathscr{B} \longrightarrow \mathscr{C}$. The "comma category" (F, G) is defined in the following way.*

(1) The objects of (F, G) are the triples (A, f, B) where $A \in \mathscr{A}$, $B \in \mathscr{B}$ are objects and $f: FA \longrightarrow GB$ is a morphism of $\mathscr{C}$.

(2) A morphism of (F, G) from (A, f, B) to (A', f', B') is a pair (a, b), where a: $A \longrightarrow A'$ is a morphism of $\mathscr{A}$, b: $B \longrightarrow B$ is a morphism of $\mathscr{B}$, and $f' \circ F(a) = G(b) \circ f$ (see diagram 1.11).

(3) The composition law in (F, G) is that induced by the composition laws of $\mathscr{A}$ and $\mathscr{B}$, thus

$$(a', b') \circ (a, b) = (a' \circ a, b' \circ b).$$

$$\begin{tikzcd}
(F,G) \arrow[r, "V"] \arrow[d, "U"'] \arrow[dr, phantom, "\alpha \nearrow"] & \mathscr{B} \arrow[d, "G"] \\
\mathscr{A} \arrow[r, "F"'] & \mathscr{C}
\end{tikzcd}$$

Diagram 1.12

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "V'"] \arrow[d, "U'"'] \arrow[dr, phantom, "\alpha' \nearrow"] & \mathscr{B} \arrow[d, "G"] \\
\mathscr{A} \arrow[r, "F"'] & \mathscr{C}
\end{tikzcd}$$

Diagram 1.13

Proposition 1.6.2 *Consider functors $F: \mathscr{A} \longrightarrow \mathscr{C}$, $G: \mathscr{B} \longrightarrow \mathscr{C}$ and their corresponding comma-category (F,G). There are two functors $U: (F,G) \longrightarrow \mathscr{A}$, $V: (F,G) \longrightarrow \mathscr{B}$ (see diagram 1.12); moreover there exists a canonical natural transformation*

$$\alpha: F \circ U \Rightarrow G \circ V.$$

Proof With the notation of 1.6.1 it suffices to define

$$U(A,f,B) = A, \quad V(A,f,B) = B,$$
$$U(a,b) = a, \quad V(a,b) = b.$$

The equality $F \circ U = G \circ V$ has no reason at all to hold in general. The natural transformation α is easily defined by $\alpha_{(A,f,B)} = f$; the fact that it is a natural transformation is just condition 1.6.1.(2). $\square$

Proposition 1.6.3 *In the situation and with the notations of 1.6.2, consider a category $\mathscr{D}$, two functors $U': \mathscr{D} \longrightarrow \mathscr{A}$, $V': \mathscr{D} \longrightarrow \mathscr{B}$ (see diagram 1.13) and a natural transformation*

$$\alpha': F \circ U' \Rightarrow G \circ V'.$$

In that case there exists a unique functor $W: \mathscr{D} \longrightarrow (F,G)$ such that

$$U \circ W = U', \quad V \circ W = V', \quad \alpha * W = \alpha'.$$

Proof The conditions imposed on W indicate immediately what it should be:

$$W(D) = (U'D, \alpha'_D, V'D)$$

for an object $D \in \mathscr{D}$ and

$$W(d) = (U'd, V'd)$$

for a morphism d of $\mathscr{D}$, which already proves the uniqueness of such a W. To prove the existence, it suffices to observe that the previous formulas indeed define a functor $W: \mathscr{D} \longrightarrow (F, G)$. $\square$

We shall refer to proposition 1.6.3 as the "universal property" of the comma category.

A special but very important case of a comma category is the "category of elements" of a functor $F: \mathscr{A} \longrightarrow \mathsf{Set}$.

Definition 1.6.4 *Consider a functor $F: \mathscr{A} \longrightarrow \mathsf{Set}$ from a category $\mathscr{A}$ to the category of sets. The category $\mathsf{Elts}(F)$ of "elements of F" is defined in the following way.*

(1) The objects of $\mathsf{Elts}(F)$ are the pairs (A, a) where $A \in |\mathscr{A}|$ is an object and $a \in FA$.
(2) A morphism $f: (A, a) \longrightarrow (B, b)$ of $\mathsf{Elts}(F)$ is an arrow $f: A \longrightarrow B$ of A such that $Ff(a) = b$.
(3) The composition of $\mathsf{Elts}(F)$ is that induced by the composition of $\mathscr{A}$.

Let us write $\mathbf{1}$ for the discrete category with a single object $\star$;

$$1: \mathbf{1} \longrightarrow \mathsf{Set}, \quad \star \mapsto \{*\}$$

is the functor which maps the unique object $\star$ of $\mathbf{1}$ to the singleton $\{*\}$. In other words, we view $\mathbf{1}$ as the full subcategory of Set generated by a singleton set. Since an element $a \in FA$ can be seen as a morphism from a singleton to FA, thus as a morphism of the type $1(A) \longrightarrow F(A)$ in Set, the category $\mathsf{Elts}(F)$ is exactly the comma category $(1, F)$. Notice that the forgetful functor $\phi_F: \mathsf{Elts}(F) \longrightarrow \mathscr{A}$ is defined by $\phi_F(A, a) = A$ on the objects and by $\phi_F(f) = f$ on the morphisms.

Another interesting example of a comma category is the "product" of two categories.

Definition 1.6.5 *The product of two categories $\mathscr{A}$ and $\mathscr{B}$ is the category $\mathscr{A} \times \mathscr{B}$ defined in the following way.*

(1) The objects of $\mathscr{A} \times \mathscr{B}$ are the pairs (A, B) with $A \in |\mathscr{A}|$, $B \in |\mathscr{B}|$ objects of $\mathscr{A}$, $\mathscr{B}$.

(2) *The morphisms* $(A,B) \longrightarrow (A',B')$ *of* $\mathscr{A} \times \mathscr{B}$ *are the pairs* (a,b) *where* $a: A \longrightarrow A'$ *is a morphism of* $\mathscr{A}$ *and* $b: B \longrightarrow B'$ *is a morphism of* $\mathscr{B}$.

(3) *The composition in* $\mathscr{A} \times \mathscr{B}$ *is that induced by the compositions of* $\mathscr{A}$ *and* $\mathscr{B}$*, namely*

$$(a',b') \circ (a,b) = (a' \circ a, b' \circ b).$$

With the product $\mathscr{A} \times \mathscr{B}$ are associated the two "projection" functors

$$p_A: \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{A}, \quad p_B: \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{B}$$

defined by the formulas

$$p_{\mathscr{A}}(A,B) = A, \quad p_{\mathscr{B}}(A,B) = B,$$
$$p_{\mathscr{A}}(a,b) = a, \quad p_{\mathscr{B}}(a,b) = b.$$

These data satisfy the following "universal property".

Proposition 1.6.6 *Consider two categories* $\mathscr{A}$ *and* $\mathscr{B}$*. For every category* $\mathscr{D}$ *and every pair of functors* $F: \mathscr{D} \longrightarrow \mathscr{A}$, $G: \mathscr{D} \longrightarrow \mathscr{B}$*, there exists a unique functor* $H: \mathscr{D} \longrightarrow \mathscr{A} \times \mathscr{B}$ *such that* $p_{\mathscr{A}} \circ H = F$, $p_{\mathscr{B}} \circ H = G$.

Proof H is the functor defined by

$$H(D) = (FD, GD) \text{ for an object } D \text{ of } \mathscr{D},$$
$$H(d) = (Fd, Gd) \text{ for a morphism } d \text{ of } \mathscr{D}. \qquad \square$$

Let us now observe the existence of a unique functor $\Delta_{\mathscr{A}}: \mathscr{A} \longrightarrow \mathbf{1}$: this is the "constant functor" to the unique object of $\mathbf{1}$ (see 1.2.8.e). Since $\mathbf{1}$ has just one single mapping, the comma category $(\Delta_{\mathscr{A}}, \Delta_{\mathscr{B}})$ is isomorphic to the product category $\mathscr{A} \times \mathscr{B}$. Proposition 1.6.6 is then a particularization of proposition 1.6.3.

A point of terminology: a functor $F: \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{C}$ defined on the product of two categories is generally called a "bifunctor" (a functor of two "variables").

1.7 Monomorphisms

When a composition law appears in some mathematical structure, special attention is always paid to those elements which are "cancellable" or "invertible" for that composition. This section is devoted to the study of left cancellable morphisms in a category.

Definition 1.7.1 *A morphism $f: A \longrightarrow B$ in a category $\mathscr{C}$ is called a monomorphism when, for every object $C \in \mathscr{C}$ and every pair of morphisms $g, h: C \rightrightarrows A$, the following property holds:*

$$(f \circ g = f \circ h) \Rightarrow (g = h).$$

We shall generally use the symbol $f: A \rightarrowtail B$ to emphasize the fact that f is a monomorphism.

Proposition 1.7.2 *In a category $\mathscr{C}$,*

(1) every identity morphism is a monomorphism,

(2) the composite of two monomorphisms is a monomorphism,

(3) if the composite $k \circ f$ of two morphisms is a monomorphism, then f is a monomorphism.

Proof We use the notation of 1.7.1 and consider another morphism $k: B \longrightarrow D$.

(1) is obvious.

(2) If f and k are monomorphisms,

$$k \circ f \circ h = k \circ f \circ g \Rightarrow f \circ h = f \circ g \Rightarrow h = g.$$

(3) If $k \circ f$ is a monomorphism,

$$f \circ g = f \circ h \Rightarrow k \circ f \circ g = k \circ f \circ h \Rightarrow g = h. \qquad \square$$

The following terminology is rather classical.

Definition 1.7.3 *Consider two morphisms $f: A \longrightarrow B$ and $g: B \longrightarrow A$ in a category. When $g \circ f = 1_A$, f is called a section of g, g is called a retraction of f and A is called a retract of B.*

Proposition 1.7.4 *In a category, every section is a monomorphism.*

Proof By 1.7.2.(1,3). $\square$

Let us now say a word about the effect of a functor on a monomorphism.

Definition 1.7.5 *Consider a functor $F: \mathscr{A} \longrightarrow \mathscr{B}$.*

(1) F preserves monomorphisms when, for every morphism f of $\mathscr{A}$,

$$f \text{ monomorphism } \Rightarrow Ff \text{ monomorphism.}$$

(2) F reflects monomorphisms when, for every morphism f of $\mathscr{A}$,

$$Ff \text{ monomorphism } \Rightarrow f \text{ monomorphism.}$$

Proposition 1.7.6 *A faithful functor reflects monomorphisms.*

Proof Consider a faithful functor $F: \mathscr{A} \longrightarrow \mathscr{B}$, a morphism $f: A \longrightarrow A'$ in $\mathscr{A}$, and suppose Ff is a monomorphism in $\mathscr{B}$. Choose another object $A'' \in \mathscr{A}$ and two morphisms $g, h: A'' \longrightarrow A$ in $\mathscr{A}$.

$$\begin{aligned} f \circ g = f \circ h &\Rightarrow Ff \circ Fg = Ff \circ Fh \\ &\Rightarrow Fg = Fh \\ &\Rightarrow g = h \end{aligned}$$

where the second implication holds since Ff is a monomorphism and the last one follows from the faithfulness of F. □

Examples 1.7.7

1.7.7.a In the category Set of sets and mappings, the monomorphisms are exactly the injections. Indeed, an element $a \in A$ can be viewed as a mapping $\overline{a}: \{\star\} \longrightarrow A$ from the singleton to A; therefore, given a monomorphism $f: A \longrightarrow B$ and elements $a, a' \in A$,

$$\begin{aligned} f(a) = f(a') &\Rightarrow f \circ \overline{a} = f \circ \overline{a'} \\ &\Rightarrow \overline{a} = \overline{a'} \\ &\Rightarrow a = a'. \end{aligned}$$

Conversely, if $f: A \longrightarrow B$ is injective and $g, h: C \rightrightarrows A$ are mappings such that $f \circ g = f \circ h$, then for every element $c \in C$

$$\begin{aligned} f \circ g = f \circ h &\Rightarrow f\big(g(c)\big) = f\big(h(c)\big) \\ &\Rightarrow g(c) = h(c) \end{aligned}$$

and therefore $g = h$.

1.7.7.b In the category Top of topological spaces and continuous mappings or its full subcategory Comp of compact Hausdorff spaces, the monomorphisms are exactly the continuous injections. Indeed, an element of a space A corresponds to a continuous mapping $\{\star\} \longrightarrow A$ from the singleton to A; therefore the argument of 1.7.7.a can be carried over.

1.7.7.c In the categories Gr of groups and Ab of abelian groups, the monomorphisms are exactly the injective group homomorphisms. The argument is again analogous, using now the bijective correspondence between the elements $a \in G$ of a group and the group homomorphisms $\overline{a}: \mathbb{Z} \longrightarrow G$ from the group of integers to G; we recall the correspondence:

$$\overline{a}(z) = z \cdot a, \quad a = \overline{a}(1).$$

1.7.7.d In the category Rng of commutative rings with a unit, the monomorphisms are exactly the injective ring homomorphisms. Repeat the argument using now the ring homomorphisms with domain the ring $\mathbb{Z}[X]$ of polynomials with integral coefficients: an element $r \in R$ of a ring R corresponds to the ring homomorphism $\overline{r}\colon \mathbb{Z}[X] \longrightarrow R$ mapping the polynomial $p(X)$ to $p(r)$; conversely $r = \overline{r}(X)$.

1.7.7.e In the category Mod_R of right modules on a ring R with unit, the monomorphisms are exactly the injective R-linear mappings. Use again the same argument using the R-linear mapping with domain the ring R itself: an element $m \in M$ of a R-module M corresponds to the linear mapping $\overline{m}\colon R \longrightarrow M$ mapping r to mr; conversely $m = \overline{m}(1)$.

1.7.7.f In the category Ban_1 of real Banach spaces and linear contractions, the monomorphisms are exactly the injective linear contractions. The elements of the unit ball of a Banach space B are in bijective correspondence with the linear contractions $\overline{a}\colon \mathbb{R} \longrightarrow B$; just put

$$\overline{a}(r) = ra \ , \ a = \overline{a}(1).$$

Therefore a monomorphism $f\colon B \longrightarrow B'$ is such that the implication

$$f(a) = f(a') \Rightarrow a = a'$$

holds for elements a, a' in the unit ball of B; by linearity of f, this fact extends to arbitrary elements a, $a' \in B$. The converse is once more obvious.

1.7.7.g The previous examples could give the wrong impression that, in "concrete" examples, a monomorphism is always exactly an injective morphism. This is false as shown by the following counterexamples. We give first an "algebraic" counterexample.

Consider the category Div of divisible abelian groups and group homomorphisms between them. The quotient morphism $q\colon \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ of the additive group of rational numbers by the group of integers is definitely not an injection, but it is a monomorphism in Div. Indeed, choose G a divisible group and $f, g\colon G \rightrightarrows \mathbb{Q}$ two group homomorphisms such that $q \circ f = q \circ g$. Putting $h = f - g$ we have $q \circ h = 0$ and the thesis becomes $h = 0$. Given an element $x \in G$, $h(x)$ is an integer since $q \circ h = 0$. If $h(x) \neq 0$ note that

$$h\left(\frac{x}{2h(x)}\right) = \frac{1}{2}$$

and therefore

$$(q \circ h)\left(\frac{x}{2h(x)}\right) \neq 0$$

which is a contradiction.

1.7.7.h Let us give now a "topological" counterexample. We consider the category whose objects are the pairs (X, x) where X is a connected topological space and $x \in X$ is a base point; in this category, a morphism $f: (X, x) \longrightarrow (Y, y)$ is a continuous mapping $f: X \longrightarrow Y$ which preserves the base points, i.e. such that $f(x) = y$. Let us consider the projection π of the circular helix $\mathcal{H}$ on the circle S^1,

$$\pi: (\mathcal{H}, h) \longrightarrow (S^1, s),$$

with $h \in \mathcal{H}$ and $s = \pi(h)$. If $f: (X, x) \longrightarrow (S^1, s)$ is a morphism in our category which admits a "lifting"

$$g: (X, x) \longrightarrow (\mathcal{H}, h)$$

through the projection π, that lifting is necessarily unique (see **Spanier**, page 67). But this expresses exactly the fact that π is a monomorphism.

1.8 Epimorphisms

We now turn our attention to right cancellable morphisms in a category.

Definition 1.8.1 *A morphism $f: B \longrightarrow A$ in a category $\mathscr{C}$ is called an epimorphism when, for every object $C \in \mathscr{C}$ and every pair of morphisms $g, h: A \rightrightarrows C$, the following property holds:*

$$(g \circ f = h \circ f) \Rightarrow (g = h).$$

We shall generally use the notation $f: B \twoheadrightarrow A$ to emphasize the fact that f is an epimorphism.

Proposition 1.8.2 *In a category $\mathscr{C}$,*

(1) every identity morphism is an epimorphism,
(2) the composite of two epimorphisms is an epimorphism,
(3) if the composite $f \circ k$ of two morphisms is an epimorphism, then f is an epimorphism.

Proof We use the notation of 1.8.1 and consider another morphism $k: D \longrightarrow B$.

(1) is obvious.

(2) If f and k are epimorphisms,

$$h \circ f \circ k = g \circ f \circ k \Rightarrow h \circ f = g \circ f \Rightarrow h = g.$$

(3) If $f \circ k$ is an epimorphism,

$$g \circ f = h \circ f \Rightarrow g \circ f \circ k = h \circ f \circ k \Rightarrow g = h. \qquad \square$$

Proposition 1.8.3 *In a category, every retraction is an epimorphism.*

Proof By 1.8.2.(1,3). $\square$

Transposing definition 1.7.5 to the case of epimorphisms, we obtain

Proposition 1.8.4 *A faithful functor reflects epimorphisms.*

Proof Consider a faithful functor $F: \mathscr{A} \longrightarrow \mathscr{B}$, a morphism $f: A' \longrightarrow A$ and suppose Ff is an epimorphism in $\mathscr{B}$. Choose another object $A'' \in \mathscr{A}$ and two morphisms $g, h: A \rightrightarrows A''$ in $\mathscr{A}$. Then

$$\begin{aligned} g \circ f = h \circ g &\Rightarrow Fg \circ Ff = Fh \circ Ff \\ &\Rightarrow Fg = Fh \\ &\Rightarrow g = h, \end{aligned}$$

where the second implication holds since Ff is an epimorphism and the last one follows from the faithfulness of F. $\square$

The similarity of the previous proofs with those of section 1.7 is striking: this is a special instance of the "duality principle" described in section 1.10.

Examples 1.8.5

1.8.5.a In the category Set of sets and mappings, the epimorphisms are exactly the surjective mappings. Choose $f: A \longrightarrow B$ a surjective mapping and $g, h: B \rightrightarrows C$ two mappings such that $g \circ f = h \circ f$. For every element $b \in B$, we can find an element $a \in A$ such that $f(a) = b$; therefore

$$g(b) = g\big(f(a)\big) = h\big(f(a)\big) = h(b),$$

which proves the equality $g = h$.

Conversely, if $f: A \longrightarrow B$ is an epimorphism, consider the two-element set $\{0, 1\}$ and the following mappings $g, h: B \rightrightarrows \{0, 1\}$:

$$\begin{aligned} g(b) &= 1 \text{ if } b \in f(A), \\ g(b) &= 0 \text{ if } b \notin f(A), \\ h(b) &= 1 \text{ for every } b \in B. \end{aligned}$$

Clearly $g \circ f = h \circ f$ is the constant mapping on 1; therefore $g = h$ and $f(A) = B$.

1.8.5.b In the category Top of topological spaces and continuous mappings, the epimorphisms are exactly the surjective continuous mappings. The previous proof applies when $\{0, 1\}$ is provided with the indiscrete topology.

1.8.5.c In the category Haus of Hausdorff topological spaces and continuous mappings between them, the epimorphisms are exactly the continuous mappings with a dense image. We recall that a continuous mapping $f: A \longrightarrow B$ has a dense image precisely when every element $b \in B$ is the limit of a net of elements of $f(A)$, i.e. a set of elements indexed by a filtered poset (see 2.13.1); when B is a Hausdorff space, the limit of a converging net is unique. Suppose $f: A \longrightarrow B$ has a dense image and choose $g, h: B \rightrightarrows C$ such that $g \circ f = h \circ f$. Given an element $b \in B$, choose a net $(a_i)_{i \in I}$ of elements in A such that $b = \lim f(a_i)$. By continuity of g, h we have

$$g(b) = \lim(g \circ f)(a_i), \quad h(b) = \lim(h \circ f)(a_i).$$

Since $g \circ f = h \circ f$ and the limit is unique, we conclude that $g(b) = h(b)$ and thus $g = h$.

Conversely if $f: A \longrightarrow B$ is an epimorphism, and B is not empty, A cannot be empty. Indeed if $B \amalg B$ is the space constituted by two disjoint copies of B, $B \amalg B$ is a Hausdorff space and the two canonical inclusions $i_1, i_2: B \rightrightarrows B \amalg B$ are continuous and distinct. A empty would yield $i_1 \circ f = i_2 \circ f$ and thus $i_1 = i_2$, since f is an epimorphism. Now consider the quotient of B which identifies with a single point the closure $\overline{f(A)}$ of the image of A; this is a Hausdorff space as a quotient of a Hausdorff space by a closed subspace; write $p: B \longrightarrow B/\overline{f(A)}$ for the corresponding continuous projection. Since $f(A)$ is not empty, we can consider as well the constant mapping $q: B \longrightarrow B/\overline{f(A)}$ on the equivalence class of the elements of $f(A)$. Clearly $p \circ f = q \circ f$ and therefore $p = q$, which proves the equality $\overline{f(A)} = B$.

1.8.5.d In the category Gr of groups and their homomorphisms, the epimorphisms are exactly the surjective homomorphisms. Indeed, a surjective homomorphism is clearly an epimorphism. Conversely suppose $f: A \longrightarrow B$ is an epimorphism. We can factor f through its image

$$A \longrightarrow f(A) \longrightarrow B,$$

thus through a surjection followed by an injection. By 1.8.2.(3), the injective part is an epimorphism and so the problem reduces to proving that an epimorphic inclusion is an identity.

Given two groups G, H with a common subgroup K, it is possible to construct the amalgamation of G and H over K: this is the group $G \star_K H$ of words constructed with the "letters" of G and H, the two copies of a "letter" of K being identified in $G \star_K H$. The amalgamation property for groups tells us that the two canonical morphisms

$$G \longrightarrow G \star_K H, \quad H \longrightarrow G \star_K H,$$

are injective and that two "letters" of G and H are identified in $G \star_K H$ just when they are the two copies of a "letter" in K (see **Kuroš**). If we apply that amalgamation property choosing the inclusion $f(A) \hookrightarrow B$ twice, we first deduce the equality of the two canonical inclusions

$$i_1 \colon B \longrightarrow B \star_{f(A)} B, \quad i_2 \colon B \longrightarrow B \star_{f(A)} B$$

since they coincide on $f(A)$ and $f(A) \longrightarrow B$ is an epimorphism. But then each element of B is already in $f(A)$ by the amalgamation property.

1.8.5.e Consider a ring R with unit. In the category Mod_R of right R-modules, the epimorphisms are exactly the surjective linear mappings. In particular, choosing $R = \mathbb{Z}$, the epimorphisms of the category of abelian groups are exactly the surjective homomorphisms. Again a surjective linear mapping is clearly an epimorphism. Conversely if $f \colon A \longrightarrow B$ is an epimorphism, consider both the quotient mapping and the zero mapping

$$p \colon B \longrightarrow B/f(A), \quad 0 \colon B \longrightarrow B/f(A).$$

From the equality

$$p \circ f = 0 = 0 \circ f$$

we deduce $p = 0$ and thus $B = f(A)$.

1.8.5.f The form of epimorphisms in the category of commutative rings with unit is known (see exercise 1.11.13); let us just emphasize the fact that epimorphisms of rings are not necessarily surjective. Consider the inclusion of the ring $\mathbb{Z}$ of integers in the ring $\mathbb{Q}$ of rational numbers, $i \colon \mathbb{Z} \longrightarrow \mathbb{Q}$. This is clearly not a surjection but it is an epimorphism of rings. Indeed given another ring A and two ring homomorphisms $f, g \colon \mathbb{Q} \rightrightarrows A$ which agree on the integers, we deduce first that for every integer $0 \neq z \in \mathbb{Z}$, z is invertible in $\mathbb{Q}$ and therefore $f(z)$ and $g(z)$ are invertible in A; clearly

$$\frac{1}{f(z)} = f\left(\frac{1}{z}\right) \text{ and } \frac{1}{g(z)} = g\left(\frac{1}{z}\right).$$

Since f and g agree on the integers, $f\left(\frac{1}{z}\right) = g\left(\frac{1}{z}\right)$ and finally,

$$\begin{aligned} f\left(\frac{z'}{z}\right) &= f\left(z' \cdot \frac{1}{z}\right) = f(z') \cdot f\left(\frac{1}{z}\right) \\ &= g(z') \cdot g\left(\frac{1}{z}\right) = g\left(z' \cdot \frac{1}{z}\right) = g\left(\frac{z'}{z}\right). \end{aligned}$$

1.8.5.g In the category Ban_1 of Banach spaces and linear contractions, the epimorphisms are the linear contractions with dense image. Choose $f\colon A \longrightarrow B$ with a dense image and $g, h\colon B \rightrightarrows C$ such that $g \circ f = h \circ f$. Since g and h agree on $f(A)$, by continuity g, h agree on on $\overline{f(A)} = B$ as well; therefore $g = h$. Conversely if $f\colon A \longrightarrow B$ is an epimorphism, the quotient of B by the closed subspace $\overline{f(A)}$ is a Banach space and both the quotient mapping p and the zero mapping are linear contractions:

$$p\colon B \longrightarrow B/\overline{f(A)}, \quad 0\colon B \longrightarrow B/\overline{f(A)}.$$

From the equalities $p \circ f = 0 = 0 \circ f$, we deduce $p = 0$ and thus $B = \overline{f(A)}$.

1.9 Isomorphisms

We consider finally the case of those morphisms of a category which are invertible.

Definition 1.9.1 *A morphism $f\colon A \longrightarrow B$ in a category $\mathscr{C}$ is called an isomorphism when there exists a morphism $g\colon B \longrightarrow A$ of $\mathscr{C}$ which satisfies the relations*

$$f \circ g = 1_B, \quad g \circ f = 1_A.$$

Clearly such a morphism g is necessarily unique; indeed if $h\colon B \longrightarrow A$ is another morphism with the same properties

$$f \circ h = 1_B, \quad h \circ f = 1_A,$$

we conclude that

$$g = g \circ 1_B = g \circ f \circ h = 1_A \circ h = h.$$

Therefore we shall call such a morphism g "the" inverse of f and we shall denote it by f^{-1}.

Proposition 1.9.2 *In a category,*

(1) every identity is an isomorphism,
(2) the composite of two isomorphisms is an isomorphism,
(3) an isomorphism is both a monomorphism and an epimorphism.

Proof

(1) is obvious.
(2) If $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are isomorphisms, so is $g \circ f$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
(3) is just the conjunction of 1.7.4 and 1.8.3 □

Proposition 1.9.3 *In a category, if a section is an epimorphism, it is an isomorphism.*

Proof If $g \circ f = 1_A$ and $f: A \longrightarrow B$ is an epimorphism, from $f \circ g \circ f = f$ we deduce $f \circ g = 1_B$. □

Proposition 1.9.4 *Every functor preserves isomorphisms.*

Proof Obvious. □

Transposing definition 1.7.5 to the case of isomorphisms, we obtain

Proposition 1.9.5 *A full and faithful functor reflects isomorphisms.*

Proof Obvious. □

Examples 1.9.6

1.9.6.a In the category Set of sets, the isomorphisms are exactly the bijections.

1.9.6.b In the category Top of topological spaces, the isomorphisms are exactly the homeomorphisms. Since a continuous bijection is in general not a homeomorphism, this provides an example where the converse of statement 1.9.2.(3) does not hold (see 1.7.7.b and 1.8.5.b).

1.9.6.c In the categories Gr of groups, Ab of abelian groups and Rng of commutative rings with unit, the isomorphisms are the bijective homomorphisms.

1.9.6.d In the category Mod_R of right modules over a ring R, the isomorphisms are the bijective R-linear mappings.

1.9.6.e In the category Ban_∞ of real Banach spaces and bounded linear mappings, the isomorphisms are the bounded linear bijections. An isomorphism is obviously bijective. Conversely if $f: A \longrightarrow B$ is a bounded linear bijection, the inverse mapping $f^{-1}: B \longrightarrow A$ is certainly linear. By the open mapping theorem, f is open because it is surjective; but "f open" means precisely "f^{-1} continuous" and thus f^{-1} is bounded.

1.9.6.f In the category Ban_1 of real Banach spaces and linear contractions, the isomorphisms are exactly the isometric bijections. An isometric bijection is obviously an isomorphism. Conversely if the linear contraction $f: A \longrightarrow B$ has an inverse mapping $f^{-1}: B \longrightarrow A$ which is also a linear contraction, then for every element $a \in A$

$$\|a\| = \|f^{-1}f(a)\| \leq \|f(a)\|$$

and thus $\|a\| = \|f(a)\|$ since f is contracting.

1.9.6.g In the category Cat of small categories and functors, the isomorphisms are those defined in 1.5.1.

1.9.6.h Going back to example 1.2.6.d, a group can be seen as a category with a single object all of whose morphisms are isomorphisms.

1.10 The duality principle

At this point the reader will have noticed that every result proved for covariant functors has its counterpart for contravariant functors and every result proved for monomorphisms has its counterpart for epimorphisms. These facts are just special instances of a very general principle.

Definition 1.10.1 *Given a category $\mathscr{A}$, the dual category $\mathscr{A}^*$ is defined in the following way:*

(1) $|\mathscr{A}^*| = |\mathscr{A}|$
(both categories have the same objects);

(2) for all objects A, B of $\mathscr{A}^*$, $\mathscr{A}^*(A, B) = \mathscr{A}(B, A)$ (the morphisms of $\mathscr{A}^*$ are those of $\mathscr{A}$ "written in the reverse direction"; to avoid confusion, we shall write $f^*: A \longrightarrow B$ for the morphism of $\mathscr{A}^*$ corresponding to the morphism $f: B \longrightarrow A$ of $\mathscr{A}$);

(3) the composition law of $\mathscr{A}^*$ is given by

$$f^* \circ g^* = (g \circ f)^*.$$

Metatheorem 1.10.2 *(Duality principle) Suppose the validity, in every category, of a statement expressing the existence of some objects or morphisms or the equality of some composites. Then the "dual statement" is also valid in every category; this dual statement is obtained by reversing the direction of every arrow and replacing every composite $f \circ g$ by the composite $g \circ f$.*

Proof If S denotes the given statement and S^* denotes its dual statement, proving the statement S^* in a category $\mathscr{A}$ is equivalent to proving the statement S in the category $\mathscr{A}^*$, and this is supposed to be valid. $\square$

For example, the notion of $f: A \longrightarrow B$ being a monomorphism in $\mathscr{A}$ means

$$\forall C \in \mathscr{A} \;\; \forall g, h \in \mathscr{A}(C, A) \;\; f \circ g = f \circ h \Rightarrow g = h.$$

The dual notion is thus that of a morphism $f: B \longrightarrow A$ which satisfies

$$\forall C \in \mathscr{A} \;\; \forall g, h \in \mathscr{A}(A, C) \;\; g \circ f = h \circ f \Rightarrow g = h$$

... which is exactly the notion of an epimorphism. With that remark in mind, it is obvious that all the results of section 1.8 are just the dual statements of the results of section 1.7: so, formally, the validity of the latter follows at once from the validity of the former via the duality principle.

The case of contravariant functors can also be reduced to the case of covariant functors via the consideration of the dual category: a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$ is just a covariant functor from $\mathscr{A}^*$ to $\mathscr{B}$ (or, equivalently, a covariant functor from $\mathscr{A}$ to $\mathscr{B}^*$).

It is interesting to notice that, in category theory, some notions are their own dual. For example $f: A \longrightarrow B$ is an isomorphism when

$$\exists g: B \longrightarrow A \quad g \circ f = 1_A, \quad f \circ g = 1_B.$$

The dual notion is that of a morphism $f: B \longrightarrow A$ with the property

$$\exists g: A \longrightarrow B \quad f \circ g = 1_A, \quad g \circ f = 1_B$$

... but this is again the definition of f being an isomorphism.

Examples 1.10.3

1.10.3.a With every category $\mathscr{A}$ we can associate a bifunctor, still written $\mathscr{A}$,

$$\mathscr{A}: \mathscr{A}^* \times \mathscr{A} \longrightarrow \mathsf{Set},$$

defined by the following formulas:

- $\mathscr{A}(A, B)$ is the set of morphisms from A to B;
- if $f: A' \longrightarrow A$ and $g: B \longrightarrow B'$ are morphisms of $\mathscr{A}$,

$$\mathscr{A}(f, g): \mathscr{A}(A, B) \longrightarrow \mathscr{A}(A', B'), \quad \mathscr{A}(f, g)(h) = g \circ h \circ f.$$

Fixing the first variable A we obtain the covariant functor defined in 1.2.8.d and fixing the second variable B we obtain the contravariant functor defined in 1.4.3.b. The bifunctor $\mathscr{A}$ is called the "Hom-functor" of the category $\mathscr{A}$ (from "homomorphism"); it is "contravariant in the first variable and covariant in the second variable".

1.10.3.b The dual of the category of sets and mappings is equivalent to the category of complete atomic boolean algebras and $(\vee - \wedge)$-preserving homomorphisms. Indeed, writing CBA for the second category, the contravariant power set functor can be seen as a contravariant functor $\mathcal{P}^*: \mathsf{Set} \longrightarrow \mathsf{CBA}$. It is well-known that every complete atomic boolean algebra B is isomorphic to the power-set $\mathcal{P}X$ of its set X of atoms. Let us prove now that $\mathcal{P}^*$ is a full and faithful functor. Given two sets X and Y, the mapping

$$\mathsf{Set}(X,Y) \longrightarrow \mathsf{CBA}(\mathcal{P}^*Y, \mathrm{P}^*X), \quad f \mapsto f^{-1}$$

is obviously injective. To prove it is surjective, let us consider a morphism $g: \mathcal{P}^*Y \longrightarrow \mathcal{P}^*X$ in CBA and an element $x \in X = g(Y)$ (g preserves the top element). Now Y is the union of its singletons and g preserves unions, so there exists some $y \in Y$ such that $x \in g(\{y\})$. Such an element y is necessarily unique since $x \in g(\{y'\})$ with $y' \neq y$ would imply

$$x \in g(\{y\} \cap \{y'\}) = g(\emptyset) = \emptyset,$$

because g preserves intersections and the bottom element. Writing $f(x)$ for that element y, it follows easily that g is just f^{-1}.

1.10.3.c The dual of the category of abelian groups and their homomorphisms is equivalent to the category of compact abelian groups and continuous homomorphisms. This is just the Pontryagin duality theorem: with every abelian group A is associated its group of characters $\hat{A} =$ Hom (A, U) where U is the circle group and the topology of $\hat{A}$ is that induced by the product topology U^A; with every homomorphism $f: A \longrightarrow B$ is associated the morphism $\hat{f}: \hat{B} \longrightarrow \hat{A}$ of composition with f.

1.10.3.d The category of finite abelian groups and their homomorphisms is equivalent to its own dual category. Indeed, it suffices to particularize the Pontryagin duality to the case of finite groups: when A is finite, $\hat{A}$ is isomorphic to A as a group and therefore is finite. But the finite compact groups are just the finite discrete groups, thus finally just the finite groups.

1.11 Exercises

1.11.1 If two ordered sets A, B are viewed as categories (see 1.2.6.b), prove that a functor from A to B is just an order preserving mapping. If $f, g: A \rightrightarrows B$ are two such functors, prove that there exists a (single) natural transformation from f to g if and only if for every element $a \in A$, $f(a) \leq g(a)$.

1.11.2 If two monoids M and N are viewed as categories (see 1.2.6.d), prove that a functor from M to N is just an homomorphism of monoids. What is a natural transformation between two such functors?

1.11.3 In exercise 1.11.2, if M and N are groups, show the existence of a natural transformation between two functors $f, g: M \rightrightarrows N$ if and only if f and g are conjugate:

$$\exists n \in N \ \ \forall m \in M \ \ f(m) = n^{-1} \cdot g(m) \circ n.$$

1.11.4 If G is a group considered as a category (see 1.9.6.h), prove that a natural transformation on the identity functor of G is just an element of the centre of G.

1.11.5 Prove that a covariant representable functor preserves monomorphisms.

1.11.6 Prove that a contravariant representable functor maps an epimorphism to a monomorphism.

1.11.7 Prove that the forgetful functor $\mathsf{Rng} \longrightarrow \mathsf{Set}$ which maps a ring to its underlying set is faithful and representable by the ring $\mathbb{Z}[X]$, but does not preserve epimorphisms. [Hint: see 1.8.5.f.]

1.11.8 If $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are small categories, prove the isomorphism of categories

$$\mathsf{Fun}(\mathscr{A} \times \mathscr{B}, \mathscr{C}) \cong \mathsf{Fun}\big(\mathscr{A}, \mathsf{Fun}(\mathscr{B}, \mathscr{C})\big),$$

where Fun denotes the category of functors and natural transformations.

1.11.9 Prove that a retraction which is a monomorphism is necessarily an isomorphism.

1.11.10 Determine the nature of the monomorphisms, epimorphisms and isomorphisms in examples 1.2.7.

1.11.11 Consider a small category $\mathscr{A}$ and the corresponding functor category $\mathsf{Fun}(\mathscr{A}, \mathsf{Set})$. Prove that a morphism α of $\mathsf{Fun}(\mathscr{A}, \mathsf{Set})$ (a natural transformation) is a monomorphism if and only if each component α_A, $A \in \mathscr{A}$, is a monomorphism in Set. [Hint: use the Yoneda lemma].

$$\mathscr{A} \quad : \quad X \xrightarrow{\;x\;} Y$$

$$\mathscr{B} \quad : \quad \begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[dr, "k"'] & B \arrow[d, "f"] \\ & C \end{tikzcd}$$

Diagram 1.14

1.11.12 The statement in 1.11.11 is no longer valid when Set is replaced by an arbitrary category $\mathscr{B}$. Consider the categories of diagram 1.14 (as a convention, identity arrows are not shown) where, in $\mathscr{B}$, the two composites $f \circ g$ and $f \circ h$ are equal to k. The category $\mathsf{Fun}(\mathscr{A}, \mathscr{B})$ is the category of arrows of $\mathscr{B}$ (see 1.2.7.c). The pair $(1_B, f)$: $(B, 1_B, B) \longrightarrow (B, f, C)$ is a monomorphism in $\mathsf{Fun}(\mathscr{A}, \mathscr{B})$ while f is not a monomorphism in $\mathscr{B}$.

1.11.13 Consider the category Rng of commutative rings with unit. A morphism $f: A \longrightarrow B$ is an epimorphism precisely when given any element $b \in B$, the equality $1 \otimes b = b \otimes 1$ holds in $B \otimes_A B$. This is also equivalent to saying that the morphism $B \longrightarrow B \otimes_A B$ is surjective, or again equivalently is an epimorphism.