CATEGORY THEORY JUNIOR PAPER SUGGESTIONS

This tutorial can potential provide the basis for a paper which will fulfill the Junior Paper requirement for math concentrators at Harvard. Here are some suggestions for Junior Paper topics (you should feel free to modify any of those listed below, or come up with your own idea). If you'd like to talk about paper topics (on this list or otherwise), let me know. I'll be around for much of the remainder of the summer and, of course, in the fall; so if you'd like to talk while you're working on your paper just send me an e-mail.

(1) **Work out the main constructions and theorems from class for a specific class of categories.** For example:
  (a) You might consider the categories associated to partially ordered sets and work out explicitly what limits, colimits, adjunctions (between posets), monad and monoidal structures, etc. are in these cases. Prerequisites beyond course material: minimal.
  (b) You might look at different categories of spaces (all spaces, compactly generated spaces, compact Hausdorff spaces) in greater depth, understand topological structures on things like the limit/colimit of underlying diagrams of sets, and work out some of the results (from Mac Lane or other sources) about standard adjunctions arising from topology. There are somewhat interesting point-set topology issues that arise, making things work in certain categories of spaces but not in others. (Mac Lane includes quite a few sections related to this material, only a few of which we briefly discussed. See: Mac Lane 2 ed, V.9, VI.9 VII.8, VII.9, for example.) Prerequisites beyond course material: knowledge of point set topology.

(2) **The Kleisli Category.** In class, we showed that every monad T in a category X arises from an adjunction $F^T \dashv G^T \colon X \to X^T$ where X^T is the category of T-algebras in X. This category has the "universal property" that, given any category A and adjunction $F \dashv G \colon X \to A$ with associated monad T, there exists a unique comparison functor (i.e. a functor such that....) from A to X^T. Thus, X^T is in some sense the "final" category receiving an adjunction from X with associated monad T. We might instead look for an "initial" category as such; this can be realized via the construction of the *Kleisli category* of a monad, which is, roughly, the category of *free* T-algebra objects in X.

The goal of this project would be to understand the construction of the Kleisli category of a monad T in X, prove that it receives an adjunction from X with monad T, and show that it has the "universal property" described above. This is all done in Mac Lane VI.5. Additionally, I'm told that the Kleisli category is the preferred way to think about monads in CS. Understanding and explaining this would be a great way to finish the paper (I don't know about this, but I'd be interested to hear if you learn about it!). Prerequisites beyond course material: none, but maybe some background in CS if you want to get into applications.

(3) **Reflective Subcategories.** We talked briefly about the notion of a reflective subcategory in class. The goal of this project would be to understand this particular type of adjunction, and go through a number of important examples that arise in applications (in math). A good starting place would be Mac Lane IV.3. Prerequisites beyond course material: none.

(4) **Different versions of Beck's Theorem.** In class, we stated conditions for an adjunction to me monadic. There are a number of similar statements where various hypotheses or conclusions of Beck's theorem are modified. Many of these are stated as exercises in Mac Lane VI.7. The goal of this project would be to understand how varying hypotheses lead to statements similar to Beck's theorem (and, in particular, to really understand the proof of Beck's theorem and how it needs to be modified to prove different similar statements). This project might be somewhat frustrating in that other textbooks use different definitions for "monadic" and "creating (limits, colimits, certain diagram shapes)", but getting these things straight would be worthwhile (and essentially you'd see that for any choice of definitions there's something like Beck's theorem which is true.) Prerequisites beyond course material: none.

(5) **Foundations.** Throughout the course, we encountered various "size" problems, where we had to be a little bit careful because "things might not be (small) sets". We only touched briefly on how paradoxes can be avoided. There are a few different resolutions to these sorts of problems. The idea for this paper would be to understand in some detail how one of these works out, and possibly survey some of the others. A starting point for this project might be section I.6 in Mac Lane, and any references Mac Lane gives. Prerequisites beyond course material: some background in logic might help.

(6) **Applications of Category Theory.** Category theory has applications in things like CS (see Kleisli category topic above). I don't know much about this stuff, but if you're interesting in finding out I'd encourage you to google "applied category theory" and see if anything that comes up strikes your fancy! Prerequisites beyond course material: Unclear. This one is very open ended!

(7) **Kan Extensions.** Kane extensions provide a general framework that subsumes many familiar concepts (e.g. limits and colimits). We'll probably talk about them a little in class, but won't have time to go into too much detail. The idea for this paper would be to really understand Kan extensions through a number of examples (both abstract – e.g. "this Kan extension realizes the notion of limit" – and concrete – i.e. "this Kan extension recovers this familiar mathematical construction".) Kan extensions are covered in detail in a number of texts, including Mac Lane's (Chapter X) and Riehl's

books, so it wouldn't be hard to find references for this project. Prerequisites beyond course material: none.

(8) **Homology and Cohomology; representability of cohomology.** Understand homology and cohomology in algebraic topology from a categorical point of view. Show that the cohomology functor $H^n(-;G)\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ is represented by *Eilenberg-Mac Lane space* $K(G,n)$. I'd suggest looking at Mac Lane VII.6 as one reference for a more abstract view on homology. For representability, you'll probably want to google or look at an algebraic topology textbook. Prerequisites beyond course material: algebraic topology (homolgoy, cohomology, basic constructions on spaces).

(9) **Representability theorems.** (Related to the previous topic.) Brown's representability theorem gives conditions under which a functor from spaces to sets is representable. This is related to the representability part of the previous topic, as it in particular implies cohomology is representable. This topic would differ from the previous one, however, since the previous one focuses on understanding Eilenberg-Mac Lane spaces, while this one would focus on proving the general representability result (thus would be more abstract). I'd suggest looking at Brown's original paper for starters, and if you find it hard to read looking elsewhere. Prerequisites beyond course material: algebraic topology (homology, cohomology, basic constructions on spaces).

(10) **Other chapters in Mac Lane.** You could also read (and from this reading move on to other sources – feel free to ask me for suggestions) some of the chapters we didn't get to in Mac Lane: chapter VIII on Abelian Categories, chapter XI on Symmetry and Braiding in Monoidal Categories...