SUMMER 2018 TUTORIAL: CATEGORY THEORY

> *"Category" has been defined in order to be able to define "functor," and "functor" has been defined in order to be able to define "natural transformation."*
>
> - Mac Lane Chapter 1.

For the first part of the course, where all material is covered thoroughly in multiple references, I'll provide a very rough outline of what is covered in class. The idea is that you should be able to read over the outline, understand all the definitions and give examples of each, and have a rough idea of how to prove the theorems.

Notational Conventions: Since we're using a few different texts, here's a (probably incomplete) summary of our notational conventions.

(1) In class, we defined a "category" to refer to what some call a "locally small category." That is, we require arrows between any two fixed objects to form a set. In practice, one also wants to consider categories where the arrows between two fixed objects aren't necessarily a set (e.g. functor categories of locally small categories), and in such cases foundational issues can be resolved. We agree to turn a blind eye as is necessary.

(2) I was somewhat inconsistent with my notation during lecture. To clarify: we will use $\mathbb{1}$, $\mathbb{2}, \dots, \mathbb{n}$ for the category associated to the poset $\{0, 1, \dots, n-1\}$. (So the category $\mathbb{2}$, for example, has two objects 0 and 1 and only one nonidentity morphism from 0 to 1). The poset $\{0, 1, \dots, n\}$ is often called $[n]$, which is confusing since the numbering doesn't match up.

(3) We agree that, for a category $\mathcal{C}$, $x \in \mathcal{C}$, $x \in |\mathcal{C}|$ and $x \in \mathrm{ob}(\mathcal{C})$ all mean that x is an object of $\mathcal{C}$. I'll try to use $x \in |\mathcal{C}|$, but may occasionally use the others.

(4) Differing from Mac Lane, we use Δ_+ for the category of finite linearly ordered sets, which includes the empty set (he calls this category Δ). We write $n + 1 = [n] = \{0, 1, \dots, n\}$. Both notations have advantages. In the world of cardinal arithmetic, $n+1$ is the more natural name for this set, since it has $n+1$ elements. Moreover, with respect to the monoidal structure on Δ_+, we get the nice fact that the linearly ordered set corresponding to the monoidal product ("sum") of n and m is the set corresponding to $n + m$. However, in topology it's more usual to call this set $[n]$, since it corresponds to an n-dimensional simplex. (Also see note (2) above for another alternative notation... Sorry!)

We will also use Δ for the category of finite, nonempty totally ordered sets, since this is standard in topology.

§1. Lecture 6/18/2018

§1.1. **Summary.**

- Material from Mac Lane Chapter 1, Riehl Chapter 1 and 2.1, Leinster Chapter 1, 3, 4.1.
- Category, functor, natural transformation.
- Key examples:
  (1) Partially ordered sets and their categories. The categories $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$ associated to finite totally ordered sets.
  (2) Set: the category of sets and set morphisms.
  (3) Algebraic examples: Grp, the category of groups and group homomorphisms; Ab, the category of abelian groups; Ring, the category of unital rings and ring homomorphisms; Vec_k, the category of vector spaces over a fixed field k and linear maps; Mod_R, the category of modules over a fixed ring R and R-module homomorphisms...
  (4) Top, Top$*$: the categories of topological spaces and continuous maps, and pointed topological spaces and basepoint-preserving continuous maps, respectively.
  (5) Functor categories.
  (6) The fundamental groupoid of a topological space.
  (7) The category associated to a partially ordered set.
  (8) The one-object category associated to a group.
  (9) The fundamental group and fundamental groupoid functors, from $\mathrm{Top}_* \to \mathrm{Grp}$ and $\mathrm{Top} \to \mathrm{Cat}$, respectively.
  (10) Forgetful functors, free functors.
  (11) (Co)homology functors from $\mathrm{Top} \to \mathrm{Ab}$.

§1.2. **Exercises (Mostly from Mac Lane's book, Chapter 1).**

(1) Given any category $\mathcal{C}$, describe what it means to give a functor $\mathbf{1} \to \mathcal{C}$, $\mathbf{2} \to \mathcal{C}$, and $\mathbf{3} \to \mathcal{C}$.
(2) Show that a functor between the categories associated to two partially ordered sets is equivalent to an order-preserving function between the sets.
(3) Show that there is no functor from Grp to Ab which, on objects, takes a group to its center.
(4) Give an example of a category $\mathcal{C}$ and two functors $F, G : \mathcal{C} \to \mathcal{C}$ which are the same on objects but different on morphisms.
(5) Given a category $\mathcal{C}$ and an object $c \in |\mathcal{C}|$, the *slice category* or *over category* $\mathcal{C}/c$ has as objects morphisms in C with codomain c (i.e. a choice of $x \in |\mathcal{C}|$ and a morphism $x \to c$ in $\mathcal{C}$), and as morphisms commutative triangles: given $f\colon x \to c$ and $f'\colon x' \to c$, a morphism from f to f' in $\mathcal{C}/c$ is a morphism $g\colon x \to x'$ in $\mathcal{C}$ so that the diagram

$$\begin{tikzcd} x \arrow[rr, "g"] \arrow[dr, "f"'] & & x' \arrow[dl, "f'"] \\ & c & \end{tikzcd}$$

commutes.

(a) Use duality (or just intuition) to define an *under category* $c/\mathcal{C}$.
(b) Fixing a commutative ring R, what is another name for the under category $R/\,\mathrm{CommRing}$? (Here, CommRing is the category of commutative rings.)

(6) Come up with some new examples of categories (not discussed in class).
(7) Try to come up with a definition of "morphism between functors" (you can of course just read the book, but the point here is really to think about the concepts on your own.)

§2. Lecture 6/20/2018

§2.1. **Summary.**

- Within a category: inverse morphism, isomorphism, epimorphism, monomorphism.
- Equivalence of categories, essentially surjective functor, faithful functor, full functor.
- Universal properties (know the universal properties for key constructions).
- Groupoid, monoid, discrete category.

§2.2. **Exercises.**

(1) Find a category with an arrow that is epic and monic, but not an isomorphism.
(2) Prove that the composite of monics is monic, and the composite of epis is epi.
(3) If $g \circ f$ is monic, then f is monic. Is the same true for g? What is an analogous statement for epis?
(4) Prove the "analogous statement for epis" from the previous problem using opposite categories.
(5) If a functor T is faithful and Tf is monic, then f is also monic. Think about what similar statements might be true (or false).
(6) Let $S, T : \mathcal{C} \to P$ be functors from a category $\mathcal{C}$ to a poset category P. Show that there is a natural transformation $S \Rightarrow T$ if and only if for all $c \in \mathcal{C}$, $Sc \leq Tc$.
(7) Show that the category Vec_k of vector spaces over a field k is equivalent to the category Matr_k defined as follows: the objects of Matr_k are in bijection with the natural numbers $0, 1, 2, \ldots$, and for any k, l, the morphisms from k to l are $l \times k$ matrices with entries in k. (You might want to use (11) on this pset).
(8) A *discrete category* is one where the only arrows are identity arrows. We have a functor from Set to Cat given by considering a set as a category with objects the elements of the set. Is this a fully faithful functor?
(9) (a) A *monoid* is a category with one object. Given an example of a monoid.
(b) A *groupoid* is a category where ever morphism is invertible. Give an example of a groupoid.

(c) Give an example of a category which is both a monoid and a groupoid.

(10) (Challenging. This is exercise 5 in Mac Lane section 5; he gives a good hint.) Show that, in the category of Groups, the epimorphisms are precisely the group homomorphisms that are surjective as set maps.

(11) (Challenging.) Show that functor is an equivalence if and only if it is fully faithful and essentially surjective on objects. (We'll do this in class on Wednesday, so the idea here is just to think a bit about why it's true, although if you can write down a proof that's great as well.)

(12) Show that a functor between the one-point categories associated to two groups G and H is "the same" as a group homomorphism. More precisely, show that the functor Grp $\to$ Mon sending a group to the associated one-point category is fully faithful.

§3. Lecture 6/22/2018

§3.1. **Summary.**

- Constructions on categories: product categories, comma categories, graphs, free categories, quotient categories.
- Functor categories and Yoneda functor.

§3.2. **Exercises.**

(1) (Revision of problem incorrectly stated in class on Wednesday.) In class, we discussed the fundamental group functor π_1 and fundamental groupoid functor Π_1, and showed that we can consider both as functors $\mathrm{Top}_* \to \mathrm{Cat}$. I asked you to find a natural transformation from $\Pi_1 \Rightarrow \pi_1$. In fact, the interesting natural transformation goes the other way (from the fundamental group to the fundamental groupoid). Explain how this works.

(2) Show that for G and H groups, $B(G \times H) = BG \times BH$, where the left hand side is the one-point category of the product group $G \times H$, and the right hand side is the product of the one-point categories associated to G and H.

(3) Check that given a small category $\mathcal{C}$, the Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ given by $X \mapsto h_X := \mathcal{C}(-, X) = \hom_{\mathcal{C}}(-, X)$ defines a covariant functor.

(4) Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, we can consider two functors as follows:
   - $\mathcal{C}(-,-)\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ defined on objects by $\langle a, b\rangle \mapsto \mathcal{C}(a,b)$
   - $\mathcal{D}(F-,F-)\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ defined on objects by $\langle a, b\rangle \mapsto \mathcal{D}(Fa, Fb)$.

   (If you're unclear on what happens to morphisms, work this out. This will also clarify why we need an "op" in the domain of both functors.) Show that maps $F_{a,b}\colon \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ given by considering F on morphisms define the components of a natural transformation
$$\mathcal{C}(-,-) \Rightarrow \mathcal{D}(F-,F-).$$

(5) An *initial object* in a category $\mathcal{C}$ is an object i so that for any object $X \in \mathcal{C}$, there there is a unique arrow $i \to X$ in $\mathcal{C}$. Dually, a *terminal object* is an object t so that for any object Y, there is a unique arrow $Y \to t$ in $\mathcal{C}$. If there is a single object which is both initial and terminal, the object is called a *zero object.*
  (a) Show that the over category $C/t = (\mathcal{C} \downarrow t)$ is equivalent (and in fact isomorphic) to the category $\mathcal{C}$.
  (b) Prove a dual statement involving initial objects.

(6) (Challenging.) In class, we showed that the data of two functors $\mathcal{C} \times \mathbb{2} \to \mathcal{D}$ and a natural transformation between them is the same as the data of a functor $\mathcal{C} \times \to \mathcal{D}$. The goal of this exercise is to characterize natural *isomorphisms* of functors in an analogous way. We can define the category J to be the two-object category with one morphism in each hom set. (This forces the morphism from one object to the other to be an isomorphism.)
  (a) Draw a picture of the category J and understand what it means to give a functor from J to another category.
  (b) Show that giving two functors $\mathcal{C} \to \mathcal{D}$ and a natural isomorphism between them is the same as giving a functor from $\mathcal{C} \times J \to \mathcal{D}$.

(7) Given natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ for F, G, H functors $\mathcal{C} \to \mathcal{D}$, show that there is a "vertical composite" natural transformation $\beta \cdot \alpha\colon F \Rightarrow H$ with components $(\beta \cdot \alpha)_c = \beta_c \cdot \alpha_c$ for all $c \in \operatorname{ob}\mathcal{C}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=60, "F"] \arrow[rr, "G"' near start] \arrow[rr, bend right=60, "H"'] \arrow[rr, phantom, bend left=30, "\Downarrow\alpha"] \arrow[rr, phantom, bend right=30, "\Downarrow\beta"] & & \mathcal{D}
\end{tikzcd}$$

§4. Lecture 6/25/2018

§4.1. **Summary.**

- Universal properties, initial and terminal objects.
- Examples of representable functors.
- Statement of Yoneda lemma.

§4.2. **Exercises.**

(1) The universal property of the cartesian product of two sets A and B is often stated as follows: a product is a set C together with maps $\pi_1\colon C \to A$ and $\pi_2\colon C \to B$ so that for any set D and any given maps $f\colon D \to A$ and $f'\colon D \to B$, there exists a unique map $F\colon D \to C$ so that $\pi_1 F = f$ and $\pi_2 F = F'$.
  (a) Find an auxiliary category $\mathcal{C}$ so the data of a product is the same as an initial object of $\mathcal{C}$.
  (b) State the universal property as an isomorphism of representable (or corepresentable) functors.

(2) If you've seen some other mathematical objects defined by universal properties (e.g. quotient group, free product of groups, tensor product of vector spaces), try to restate these universal properties as (1) being initial or final in an ambient category; (2) representing some functor.

(3) Find a representing object for the forgetful functor Ring $\to$ Set (which takes a ring to its underlying set, and a ring map $R \to R'$ to itself viewed as a set map).

(4) (Challenging.) On the first day of class, we talked about a covariant power set functor $\mathcal{P}\colon \mathrm{Set} \to \mathrm{Set}$, given on objects by

$$X \mapsto \mathcal{P}(X)$$

and on morphisms by

$$(f\colon X \to Y) \mapsto (a \subset X \mapsto f(a) \subset Y).$$

Alternatively, we can define a contravariant power set functor $\tilde{\mathcal{P}}\colon \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ on objects by

$$X \mapsto \mathcal{P}(X)$$

(that is, it's the same on objects: it takes a set and returns its power set), and on morphisms by

$$(f\colon X \to Y) \in \mathrm{Set}(X, Y) \mapsto f^{-1} \in \mathrm{Set}(\mathcal{P}(Y), \mathcal{P}(X))$$

where f^{-1} is the preimage function defined by

$$a \subset Y \mapsto f^{-1}(a) \subset X.$$

Show that the functor $\tilde{\mathcal{P}}$ is corepresentable and describe a representing object.

(5) Show that a representable functor $F\colon \mathcal{C} \to \mathrm{Set}$ preserve monomorphisms in the following sense: if $f\colon a \to b$ is a monomorphism in $\mathcal{C}$, then Ff is a monomorphism in Set. Use the contrapositive to give examples of functors that are *not* representable.

(6) (Understanding the statement of the Yoneda lemma). We'll do this in class on Wednesday, but it's worthwhile to work it out for yourself.
   (a) Write down explicitly the "naturality" part of the statement of the Yoneda lemma given in class today.
   (b) Write down a contravariant version of the Yoneda lemma (i.e. a statement for functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$).

§5. Lecture 6/27/2018

§5.1. **Summary.**

- Yoneda lemma and proof, continued.
- Corollaries and examples.
- Category of elements: going between "initial objects" and representing functors.

§5.2. **Exercises.**

(1) Prove the Yoneda lemma. We sketched this in class, so you should write down all the details explicitly:
   (a) Show that the map $\mathrm{Nat}(\mathcal{C}(c,-),F) \to Fc$ given by
   $$(\alpha\colon \mathcal{C}(c,-) \Rightarrow F) \mapsto \alpha_c(\mathrm{Id}_c)$$
   is a bijection by constructing a map in the other direction and checking the constructions are mutually inverse.
   (b) Show that the correspondence is natural in $c \in \mathcal{C}$ and $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$.

   If this is too easy for you, try to work out the contravariant version (i.e. the version for functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. If you're finding this hard, try to do 6(a) from the previous exercise set.

(2) Explicitly describe the category of elements for the functor $\mathcal{C} \to \mathrm{Set}$ which sends every object of $\mathcal{C}$ to a fixed one-object set and every arrow to the identity of that set. Describe initial and terminal objects in the category of elements.

(3) Describe a universal property of the cartesian product of two sets, in the sense we described in class today: a universal property of an object $c \in \mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathrm{Set}$ (or $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$) together with $x \in Fc$ so that x gives a natural isomorphism of $\mathcal{C}(c,-)$ with F via the construction of the Yoneda lemma (or, in the case of $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, x gives a natural isomorphism of $\mathcal{C}(-,c)$ with F).

(4) Come up with a functor $F\colon \mathrm{Vect}_k \to \mathrm{Set}$ which is represented by the tensor product $V \otimes_k V'$ of two fixed vector spaces V and V'. That is, you want that
   $$F(W) \simeq \mathrm{Hom}_{\mathrm{Vect}}(V \otimes_k V', W)$$
   naturally in $W \in \mathrm{Vect}_k$. Use this to derive a universal property of the tensor product, and rephrase it to assert that $V \otimes_k V'$ together with some auxiliary data (determined by the functor you use) is an initial object of an auxiliary category.

(5) Recall that the category $\mathbb{2}$ represents the functor $F\colon \mathrm{Cat} \to \mathrm{Set}$ which takes a small category to its set of morphisms. Rephrase this representability as the statement that $\mathbb{2}$ satisfies a universal property. That is, come up with an element $x \in F(\mathbb{2})$ which, via the construction of the Yoneda lemma, gives a natural isomorphism $\mathrm{Cat}(\mathbb{2}, \mathcal{C}) \simeq \mathrm{mor}(\mathcal{C})$.

§6. Lecture 6/29/2018

§6.1. **Summary.**

- Examples of universal properties and associated representable functors.
- Diagrams in categories.
- Limits and colimits of diagrams.
- Examples (especially in Set).

§6.2. **Exercises.**

(1) Describe what it means to be a limit of a diagram indexed by the empty category (you should get something familiar!), either in an arbitrary category or in the category of sets.
(2) Explicitly describe morphisms in the category of cones over a diagram $D\colon J \to \mathcal{C}$. (We talked about this category, but I wasn't careful about describing its arrows). Do the same for the category of cocones.
(3) Go through the process of dualizing the definition of a limit to see that a colimit of a diagram is a *initial* object in the category of cocones on the diagram. (It might be helpful to observe that a cocone on a diagram $D\colon J \to \mathcal{C}$ is a cone the associated diagram $D^{\mathrm{op}}\colon J^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.)
(4) Explicitly compute the limit of a $\mathbb{Z}$-indexed diagram in the category of sets. (Start by understanding what it means to give a functor from the category associated to the poset $\mathbb{Z}$ to Set; then understand limits of such a diagrams.)
(5) Rephrase being a limit of a diagram $D\colon J \to \mathcal{C}$ as representing a functor from $\mathcal{C} \to \mathrm{Set}$. (Hint: we've already described the "category of elements" of the functor we want to represent.)
(6) Describe limits and colimits of diagrams indexed by arbitrary discrete (small) categories (categories with only identity morphisms) in the category of sets. (We did the case for a two-object discrete category.)
(7) As we've seen, certain indexing categories come up a lot, so limits over diagrams of these shapes get special names. Another "special" example of a shape for a diagram is

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] & \cdot \end{tikzcd}$$

I.e. a category J with two object and two nontrivial parallel morphisms from one object to the other. Limits of this diagram are called *equalizers*, and colimits are called *coequalizers*.
   (a) Compute the equalizer and coequalizer of an arbitrary diagram of shape J (defined above) the category of sets.
   (b) Compute the equalizer and coequalizer of the diagram in the category of groups which takes both objects to $\mathbb{Z}$, one morphism to the zero homomorphism, and the other to multiplication by n.

§7. Lecture 7/2/2018

§7.1. **Summary.**

- A bit more on colimits. Proofs from last time.
- Adjunctions. Key properties. Units and counits.
- Key examples.

§7.2. **Exercises.**

(1) (Details of example from class.) Fix $f\colon S \to R$ a ring map. We have a map $R_f\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ called *restriction of scalars*, which on objects takes an R-module M to the S-module M with the same underlying group structure, and S-action given by $s \cdot m = f(s) \cdot m$ for $m \in M$, $s \in S$ (note $f(s) \in R$, so $f(s) \cdot m$ is already defined since M is an R-module).

We can also consider *extension of scalars* $E_f\colon \mathrm{Mod}_S \to \mathrm{Mod}_R$ which is given on objects and morphisms by applying the functor $- \otimes_S R$ (noting that f gives R and S-module structure). Show that E_f is left adjoint to R_f.

(2) (Generalization of 1). Given unital (not necessarily commutative) rings R and S and an (R,S)-bimodule X, show that the functor $\mathrm{Hom}_S(X,-)\colon \mathrm{Mod}_S \to \mathrm{Mod}_R$ is right adjoint to $- \otimes_R X\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$. Explicitly, that there is a natural bijection

$$\mathrm{Hom}_S(Y \otimes_R X, Z) \simeq \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X,Z)),$$

natural in $Y \in |\,\mathrm{Mod}_R\,|$ and $Z \in |\,\mathrm{Mod}_S\,|$.

(3) For any category $\mathcal{C}$, we can consider the functor $\mathcal{C} \to \mathbb{1}$ which takes every object to the unique object of $\mathbb{1}$ and every morphism to the identity. (Recall the category $\mathbb{1}$ has one object and only the identity morhpism.) Assuming that this functor has right and left adjoints, describe them in terms of familiar definitions.

(4) Assume $\mathcal{C}$ has pullbacks and pushouts, which are limits and colimits, respectively, of diagrams indexed by

$$\text{J=} \quad \begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Describe the unit and counit of the left and right adjoint of the map $\Delta\colon \mathcal{C} \to \mathrm{Fun}(J, \mathcal{C})$ which takes an object c to the constant functor with value c.

(5) Describe a left adjoint for the inclusion $\mathrm{Ab} \to \mathrm{Grp}$ of abelian groups into all groups.

(6) (Construction/ verification.) In stating the triangle identities for the unit and counit of an adjunction, we encountered an instance of a general procedure called *whiskering*, which allows us to "compose" a natural transformation with a functor. Suppose we're given a diagram as follows

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] & \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \arrow[r, "I"] & \mathcal{D} \end{tikzcd}$$

I.e. two parallel functors $G, H\colon \mathcal{B} \to \mathcal{C}$, a functor $F\colon \mathcal{A} \to \mathcal{B}$, a functor $I\colon \mathcal{C} \to \mathcal{D}$, and a natural transformation $\alpha\colon G \Rightarrow H$. Then we can:

(a) Whisker α on the right by F to get a natural transformation $\alpha F\colon GF \Rightarrow HF$ with components defined by

$$(\alpha F)_A := \alpha_{F(A)}\colon GF(A) \to HF(A).$$

(b) Whisker α on the left by I to get a natural transformation $I\alpha\colon IG \Rightarrow IH$, with components defined by

$$(I\alpha)_A := I(\alpha_A)\colon IG(A) \to IH(A).$$

Verify that both whiskerings (a) and (b) above define natural transformations.

(7) Verify that the definition of an adjunction in terms of natural bijections on hom sets implies the triangle identities for the unit and counit.

(8) (Challenging.) Show that adjoints are unique up to unique isomorphism. That is, given $F\colon \mathcal{C} \to \mathcal{D}$, any two right adjoints are naturally isomorphic, and the isomorphism is unique if we require it to be compatible with the the adjunction. The analogous statement holds for left adjoints.

§8. Lecture 7/6/2018

§8.1. **Summary.**

- Adjoint equivalences. See handout for clarification!
- (Co)eflect subcategories: examples Ab $\subset$ Grp (reflective), Torsion $\subset$ Ab (coreflective).
- Getting new adjoints from old – functor categories (Riehl 4.4)
- Continuous, cocontinuous functors. RAPL (right adjoints preserve limits)/ LAPC (left adjoints preserve colimits).

§8.2. **Exercises.**

(1) (a) (Stated in class.) Prove that for any $\mathcal{C}$ and $c \in \mathcal{C}$, the functor $\mathcal{C}(c, -)$ preserves limits. Explicitly, given a diagram A of shape J in $\mathcal{C}$, give an isomorphism of sets

$$\lim_{i\in J} C(c, Ai) \simeq C(c, \lim_{i\in J} Ai),$$

natural in $c \in |\mathcal{C}|$.

(b) Show that if F is naturally isomorphic to G and G preserves limits of shape J, [1] the F also preserves limits of shape J.

(c) Conclude that all (covariant) representable functors preserve limits.

(d) Is there an analgous statement that is true for contravariant representable functors? (I.e. functors isomorphic to $\mathcal{C}(-, c)$.)

(2) (Generalities on whiskering.) Before attempting this problem, do problem 6 from the previous exercise set! Consider a diagram

[1] We defined preservation of limits in general. We say that a functor preserves limits *of shape* J if the limit preservation condition is held for diagrams of shape J, but not necessarily for arbitrary small diagram shapes.

$$
\begin{tikzcd}
\mathcal{A}' \arrow[r, "F'"] & \mathcal{A} \arrow[r, "F"] & \mathcal{B} \arrow[r, bend left=60, "G"] \arrow[r, "H"] \arrow[r, bend right=60, "K"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{C} \arrow[r, "I"] & \mathcal{D} \arrow[r, "I'"] & \mathcal{D}'
\end{tikzcd}
$$

This exercise justifyies omitting of parentheses in many situations. In this problem we denote composition of functors with $\circ$, vertical composition of natural transformations with $\star$, and whiskering with juxtaposition in order to avoid ambiguity. Show that:

(a) Whiskering on either side is associative: $I'(I\alpha) = (I' \circ I)\alpha$, and $(\alpha F)F' = \alpha(F \circ F')$.
(b) Whiskering on different sides commutes: $I(\alpha F) = (I\alpha)F$.
(c) Whiskering on either side is distributive: $(\beta \star \alpha)F = (\beta F) \star (\alpha F)$ and $I(\beta \star \alpha) = (I\beta) \star (I\alpha)$.

(3) (Examples: RAPL, LAPC are useful!) Verify the following statements by realizing a relevant functor as a right or left adjoint, and using RAPL or LAPC.

(a) For any sets A, B, C,

$$A \times (B \sqcup C) \simeq (A \times B) \sqcup (A \times C),$$

$$(B \times C)^A \simeq B^A \times C^A,$$ [2]

and

$$B^{(A \sqcup C)} \simeq B^A \times B^C.$$

(b) For any vector spaces U, V, W,

$$U \otimes (V \oplus W) \simeq (U \otimes V) \oplus (U \otimes W).$$

(4) For a fixed set X, so that the functor $X \times -\colon \mathrm{Set} \to \mathrm{Set}$ cannot have a *left* adjoint unless X is a one-point set.

(5) Suppose $i\colon \mathcal{D} \hookrightarrow \mathcal{C}$ is a reflective subcategory with reflector $L\colon \mathcal{C} \to \mathcal{D}$. As usual, let η and ϵ be the unit and counit of the adjunction.

(a) Show that $\eta L = L\eta$, and moreover that each is a natural isomorphism.
(b) Show that an object $c \in |\mathcal{C}|$ is in the *essential image* of i, meaning that it is isomorphic to an object in $\mathcal{D}$, if and only if η_c is an isomorphism.
(c) Show that c is in the essential image of i if and only if for all $f\colon a \to b$ in $\mathcal{C}$ so that $L(f)$ is invertible in $\mathcal{D}$, the precomposition map $f^*\colon \mathcal{C}(b, c) \to \mathcal{C}(a, c)$ is an isomorphism in Set.

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[2] Here, $X^Y = \mathrm{Set}(Y, X)$ is the set of all set maps from Y into X.

§9. Lecture 7/11/2018

§9.1. Summary.

- Completeness and cocompleteness; Set is complete and cocomplete. (Mac Lane V.2, Riehl
- Preservation (continued), reflection, and creation of limits. (Mac Lane V.1, V.4; Riehl 3.3).
- General adjoint functor theorem (Mac Lane calls this Freyd's adjoint functor theorem.)

§9.2. Exercises.

(1) Come up with an example of a category that is not complete, or not cocomplete.

(2) (Functoriality of (co)limits.) Show that if $F, G\colon J \to C$ are parallel functors, and that C has limits of shape J. Show that a natural transformation $\alpha F \Rightarrow G$ induces a map $\lim F \Rightarrow \lim G$. Show that if α is a natural isomorphism, then the map induced on limits is an isomorphism in C. (We've implicitly used this before to replace diagrams by "more convenient" ones.)

(3) Prove a dual statement to the result that a category with all small limits and all equalizers of pairs of arrows is small-complete. (The conclusion should be that under dual hypotheses, the category is small-cocomplete, meaning it has all colimits indexed by small categories.)

(4) Show Set is small-cocomplete. (Use the previous problem, or explicitly construct an arbitrary small colimit.)

(5) Show that the forgetful functor $V\colon$ CH $\to$ Set from compact hausdorff spaces to sets admits a left adjoint.

(6) (Computing limits of functors pointwise. Important but challenging.) Suppose B is small. Let $F\colon A \to C^B = \mathrm{Fun}(B, C)$ be a functor. Then, for each fixed b, we can consider the functor $F_b\colon A \to C$ given on objects by $a \mapsto (F(a))(b)$. Suppose that the functors F_b have limits L_b for each $b \in B$. Show that:
   (a) If $\tau_b\colon L_b \Rightarrow F_b$ are the limit cones, then there is a unique functor $L\colon B \to C$ given on objects by $b \mapsto L_b$ so that τ_b define components of a limit cone $L \Rightarrow F$. That is, L is a limit of the diagram F of shape A in $\mathrm{Fun}(B, C)$. Less formally, this says that if all relevant limits exist, limits in a functor category may be computed "pointwise:" $(\lim F)(b) = \lim_{a \in A}(F_b(a))$.
   (b) Use the above to show that, if a category C is has limits of shape A, so does $\mathrm{Fun}(B, C)$ for any category B.

(7) (Topological examples of adjoints.) (Some of these aren't GAFT problems. I just wanted to give some more examples from topology.)
   (a) Show that the forgetful functor $G\colon$ Top $\to$ Set has a left adjoint and a right adjoint (but the left and right are different!).
   (b) Show that inclusion functor Haus $\to$ Top of Hausdorff spaces into all spaces admits a left adjoint. (Use GAFT!) How would you describe what this adjoint does on objects?

(c) Consider the full subcategory Lconn $\subset$ Top of locally connected spaces. Let $F\colon$ Set $\to$ Lconn map a set to the discrete space with the set as its underlying set. Show that this functor admits a left adjoint C which takes a space X to its set of components. Show that C itself cannot have a left adjoint, because it does not preserve equalizers.

§10. Lecture 7/13/2018

§10.1. **Summary.**

- Review lecture!

§10.2. **Exercises.**

(1) Send me requests for solutions to the exercises! I won't write up complete solutions to all of them (or even most of them), but I'll write up 5-10 solutions for problems so far in the course, based on popular request.

§11. Lecture 7/16/2018

§11.1. **Summary.** Monads: Mac Lane VI.1-VI.4 (We'll follow this pretty closely. See pp. 124 for a discussion of type τ algebras, which he uses to motivate the chapter; understanding this is not strictly necessary, but if you're trying to read Mac Lane somewhat linearly it's helpful.)

- Monads in categories.
- Examples: Monads defined by adjunctions, monads for preorders. Free group (ring, module) monads.
- Algebras for a monad.
- All monads come from adjoints. Recovering monads from their algebras.

§11.2. **Exercises.**

(1) (Some verifications about compositions of natural transformations.)

(a) (Interchange law for horizontal and vertical compositions of natural transformations.) Given a diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=60, "F"] \arrow[r, "F'" description] \arrow[r, bend right=60, "F''"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & \mathcal{B} \arrow[r, bend left=60, "G"] \arrow[r, "G'" description] \arrow[r, bend right=60, "G''"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \mathcal{C}
\end{tikzcd}$$

Show that

$$(\beta' \cdot \beta)(\alpha' \cdot \alpha) = (\beta\alpha) \cdot (\beta'\alpha'),$$

where $\cdot$ denotes vertical composition, and juxtaposition denotes horizontal composition.

(b) (Whiskering as horizontal composition.) Show that for any $F, F'\colon \mathcal{B} \to \mathcal{C}$, any $H\colon \mathcal{A} \to \mathcal{B}$, any $G\colon \mathcal{C} \to \mathcal{D}$, and $\alpha\colon F \Rightarrow F'$, that

$$\alpha H = \alpha \mathbb{1}_H \tag{1}$$

and

$$G\alpha = \mathbb{1}_G \alpha, \tag{2}$$

where in both equations the left-hand side denotes whiskering, and the right-hand side denotes horizontal composition.

(2) A group is a group object in the category of sets. A group object in the category Grp is a group G equipped with a group homomorphism $\mu\colon G \times G \to G$, a group homomorphism e from the trivial group to G, and a group homomorphism $Inv\colon G \to G$ so that $\mu(\mu(-,-),-) = \mu(-,\mu(-,-))$, $\mu(e,-) = \mu(-,e) = \mathbb{I}_G)$, and $\mu \circ (Inv \times \mathbb{I})x = e$ for all $x \in G$. Show that a group object in Grp is an *abelian* group.

(3) (Mentioned in class without verification.) Show that, for a fixed ring R, the functor $R \otimes_{\mathbb{Z}} -\colon$ Ab $\to$ Ab is a monad. Show that algebras for this monad are precisely abelian groups with the additional structure of an R-module.

(4) (Getting R-modules in a different way). Consider the forgetful functor $G\colon R$-Mod $\to$ Set, which admits a left adjoint.
   (a) Prove that the left adjoint is given on objects by sending a set X to the set of maps $X \to R$ which are nonzero at only finitely many elements of X. Use this to explicitly describe the monad $< T_R, \eta, \mu >$ of the adjunction.
   (b) Show that $< T_R, \eta, \mu >$-modules are precisely R-modules.

(5) (a) Write down a complete definition of a category Mon(X) of monads in a fixed category X. (In particular, define the notion of a morphism of monads as a natural transformation of functors which is compatible with the multiplication and unit in an appropriate sense.)
   (b) Show that, given a natural transformation $\theta\colon T \Rightarrow T'$ of monads, there is a functor $\theta^*\colon X^{T'} \to X^T$ from T'-algebras in X to T-algebras in X.

(6) Show that, for any given monad $< T, \eta, \mu >$ on a category X, show that the right adjoint functor $G\colon X^T \to X$ (as defined in class) creates limits.

(7) Come up with a good explanation for why a monad in a preorder is called a "closure operation". (David says it's probably a discrete/ CS sort of notion. Maybe talk to him about this!)

§12. Lecture 7/18/2018

§12.1. **Summary.**

- Comparison functors, comparison theorems. Example: semigroups (Mac Lane VI.4).
- Statement of Beck's Theorem characterizing monadic adjunctions (Mac Lane VI.7).
- Split coequalizers (Mac Lane VI.6).
- Outline of proof of Beck's theorem.

§12.2. **Exercises.**

(1) (If you run into trouble, much of this is done in Mac Lane VI.6.) In class we defined a split coequalizer. This exercise builds some theory around such concepts. A *fork* in a category $\mathcal{C}$ is a special name for a cocone over a coequalizer diagram. More precisely, it is a diagram

$$\begin{tikzcd} a \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & b \arrow[r, "e"] & c \end{tikzcd}$$

in $\mathcal{C}$ so that $ef_0 = ef_1$.

(a) We say that a fork is *split* if there exists $s\colon c \to b$ and $t\colon b \to a$ so that $es = 1_c$, $f_0t = 1_b$, and $f_1t = se$. Show that a split fork is a coequalizer.

(b) Show that a split fork (which by the previous part is a coequalizer, and thus a split coequalizer) is an absolute coequalizer (as defined in lecture: the coequalizer diagram remains a coequalizer diagram after applying any functor to any category).

(2) (Equivalent conditions for being a map of adjunctions – used in the proof of the comparison theorem.)[3] Suppose we have adjunctions $< F, G, \eta, \epsilon >$ where $F\colon X \longleftrightarrow A\colon G$ with $F \dashv G$ and $< F', G', \eta', \epsilon' >$ where $F'\colon X' \longleftrightarrow A'\colon G'$ with $F' \dashv G'$. Suppose also given $H\colon X \to X'$, $K\colon A \to A'$ so that $HG = G'K$, and $F'H = KF$.

Show that the following are equivalent:

(a) For all $x \in X$ and $a \in A$, the following diagram of sets commutes:

$$\begin{tikzcd} A(Fx, a) \arrow[r, "\simeq"] \arrow[d] & X(x, Ga) \arrow[d] \\ A'(KFx, Ka) \arrow[d, "="] & X'(Hx, HGa) \arrow[d, "="] \\ A'(F'Hx, Ka) \arrow[r, "\simeq"] & X'(Hx, G'Ka) \end{tikzcd}$$

(b) $H\eta = \eta' H$.

(c) $\epsilon' K = K\epsilon$.

If any of the above equivalent conditions hold, we say that the pair (H, K) is a *map of adjunctions.*

(3) Consider the free/ forgetful adjunction between Ab and Rng[4] where the right adjoint $G\colon$ Ring $\to$ Ab forgets the multiplicative structure and simply views R as an abelian group.

(a) Give a direct description for the associated monad T in Ab, like we did in class for the monad in Set associated to the forgetful functor from semigroups. Replace $\sqcup$ by infinite direct sup of

[3]If you have problems with this, see p. 99 in (your shiny new copy of) Mac Lane! However, the proof is a bit sparse – I encourage you to fill in the details.

[4]Mac Lane uses Rng as opposed to Ring to mean unital rings. I believe I might previously have said that I would reserve Rng for nonunital rings; henceforth I'm following Mac Lane's convention. Who uses nonunital rings anyhow? (Actually not a rhetorical question... I'd like to know!)

abelian groups, and cartesian product by n-fold tensor product over $\mathbb{Z}$.

(b) Use this explicit description of T to describe T-algebras in Ab, and show that the comparison functor from Rng to T-algebras is an isomorphism.

(4) (This might be quite challenging.) Show that Beck's theorem applies to the forgetful functor R-Alg $\to$ R-Mod by verifying condition (2) or (3) in the statement of Beck's theorem from class.

§13. Lecture 7/20/2018

§13.1. **Summary.**

- Completing proof of Beck's theorem (Mac Lane VI.6)

§13.2. **Exercises.**

(1) Verify things we didn't check in class:

(a) (Used in the proof of (1) implies (2) for Beck's theorem. Generally a good fact to know!) Show that coequalizers are epimorphisms, and equalizers are monomorphisms.[5] (Hint: you actually only have to show one of these! Why?)

(b) (Used in the proof of our Special Lemma for Beck's theorem.) Let $F\colon X \to B$ with $F \dashv G$. Let $b \in |B|$. Consider the fork[6]

$$FGFGb \rightrightarrows FGb \xrightarrow{\epsilon_b} b$$

where the parallel arrows are ϵ_{FGb} and $FG\epsilon_b$ (this is the "canonical presentation" for b). Show that the diagram obtained by applying G to the diagram above is a split fork. (Hint: after applying G, you can use the triangle identities for the adjunction $F \dashv G$ to construct splittings.)

(2) As suggested in class, work out in detail what the "canonical presentation" used in the proof of our Special Lemma for Beck's theorem does in the case of the free-forgetful adjunction between Ab and Set . (See the previous problem for the definition of the canonical presentation.) If you aren't familiar with the notion of a presentation of a group, it might be helfpul to look this up.

(3) Show that if a functor $G\colon A \to X$ fits into an adjunction $F \dashv G$ which is monadic (meaning that monad T associated to the adjunction gives rise to an isomorphism $A \simeq X^T$ via the comparison functor), then G *reflects isomorphisms*: if Gf is an isomorphism in X, then f is an isomorphism in A.

---

[5]There's a good way to remember this: coequalizers are like quotient maps (think about cokernels in Grp), which are surjective, which are precisely the epimorphisms in Set and Grp; equalizers are like inclusions (think about kernels in Grp), which are injective, which are precisely monomorphisms in Set and Grp.

[6]See problem (1) in §12.2 for the definition

(4) We can prove an analogous statement to Beck's theorem as follows (one might argue this is the "more natural" version of Beck's theorem): Given an adjunction $F \dashv G$ with $F\colon X \to A$, the following are equivalent:
   (a) The comparison functor $K\colon A \to X^T$ is an **equivalence** of categories.
   (b) G admits, preserves, and reflects coequalizers for parallel pairs of arrows in A so that the parallel arrows obtained by applying G have an absolute coequalizer in X.
   (c) G admits, preserves, and reflects coequalizers for parallel pairs of arrows in A so that the parallel arrows obtained by applying G have a split coequalizer in X.

   (If you're feeling ambitious, do this in detail. Otherwise, think about it enough to understand how this statement is similar to the statement of Beck's theorem, and to have an idea how the proof of Beck's theorem must be modified.)

§14. Lecture 7/23/2018

§14.1. **Summary.**

- Monoidal categories. (Mac Lane VII.1,2,3)
- Examples of monoidal categories and the associated notion of monoid: Set, Top, endofunctors, modules, Cat....
- Category of monoidal categories; monoidal functors.
- Monoids in monoidal categories (Mac Lane VII.3).

§14.2. **Exercises.**

(1) Verify in detail that $\langle \mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z}\rangle$ is a monoidal category. How do you produce the natural isomorphisms α, λ, ρ in this case?

(2) Let $\langle B, \otimes, e, \alpha, \lambda, \rho\rangle$ be a monoidal category. Use the axioms for a monoidal category to show that for any b, c in B, the diagram

$$\begin{tikzcd} e \otimes (b \otimes c) \arrow[r, "\alpha"] \arrow[d, "\lambda"] & (e \otimes b) \otimes c \arrow[d, "\lambda\otimes 1"] \\ b \otimes c \arrow[r, "="] & b \otimes c \end{tikzcd}$$

   commutes.
   (a) Verify in detail that any category with finite products has a natural monoidal structure, with monoidal product given by the product.
   (b) Verify that the opposite category of a monoidal category has a natural monoidal structure.
   (c) Show that the category MonCat of small monoidal categories (with morphisms as defined in class) has a natural monoidal structure.
   (d) Show that for any monoidal category B, and any category C, the functor category $\mathrm{Fun}(C, B)$ is a monoidal category.

(3) For a given monoidal category B, give a definition of the category of monoid objects in B, which we denote by Mon_B. The objects should be triples $\langle c, \mu, \eta\rangle$, where c is a monoid object and μ, η, are

chosen structure maps. [7] (So your job is to define morphisms in this category.)

(4) Show, in detail, that monoid objects in the monoidal category $\langle \mathrm{Ab}, \otimes, \mathbb{Z} \rangle$ are precisely rings.

(5) Prove that if a symmetric monoidal category $\langle B, \otimes, U \rangle$ has finite products, then so does the category Mon_B of monoid objects in B.

§15. Lecture 7/25/2018

§15.1. Summary.

- Monoids in monoidal categories continued. Remarks: choices involved in giving a category with finite products a monoidal structure; canonical isomorphisms between different "bracketings of iterates of $\otimes$".
- Construction of free monoids.
- Monoid actions. (Mac Lane VII.4). Examples: monoid objects acting on themselves; actions of algebras in $\langle \mathrm{Vect}_K, \otimes_K, K \rangle$.
- The categories Δ and Δ_+. Classifying monoids in strict monoidal categories (Mac Lane VII.5).

§15.2. Exercises.

(1) Let $\langle T, \eta, \mu \rangle$ be a monoid object in the symmetric monoidal category of endofunctors $X \to X$ (a.k.a. a monad in X). Show that an endofunctor $S\colon X \to X$ is acted on by T if and only if there exists a functor $S'\colon X \to X^T$ so that $S = G^T S'$. (Where X^T is the category of algebras for the monad T in X, and G^T is as defined to realize T as the monad of an adjunction.)

(2) (a) Given a monoidal category $\langle B, \otimes, e \rangle$, a monoid $\langle c, \mu, \eta \rangle$ in B, and a, a' objects of B both equipped with left actions ν, ν' of c, give a definition of a morphism $f\colon \langle a, \nu \rangle \to \langle a', \nu' \rangle$ of actions. Use this to define a category LAct_c of objects in B upon which C acts.
(b) The forgetful functor $\mathrm{LAct}_c \to B$ has a left adjoint; take a guess at how it is defined. (See Mac Lane VII.4 if you get stuck.)

(3) (a) Show that monomorphisms in Δ are precisely the injective order-preserving maps.
(b) Show that the epimorphisms are precisely the surjective ones.

(4) (a) Explicitly write down a definition for a comonad by dualizing the definition of a monad. (Note that, if $\langle B, \otimes, e \rangle$ is monoidal, then we get a natural monoidal structure on B^{op}, as you showed in (2)(b) on the previous pset.)
(b) Give a candidate category to classify comonoids in strict monoidal categories (in the same sense that Δ classifies monoids in strict monoidal categories).

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[7] Note that we don't want the objects to be simply objects of B which admit a monoid structure – a fixed object of B could have different monoid structures, and we want these different structures on the same underlying object of B to give rise to different objects in Mon_B.

§16. Lecture 7/27/2018

§16.1. Summary.

- Geometric interpretation of Δ.
- Monoidal closure (Mac Lane VII.7).
- Compact generated spaces (Mac Lane VII.8).
- Loops and suspensions (Mac Lane VII.9).

§16.2. Exercises.

(1) Given a topological space X, verify that we get a functor
$$\mathrm{Sing}(X)\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$$
by sending the element $[n] \in \Delta$ to the set $\mathrm{Sing}(X)[n]$ of all continuous maps $\Delta[n] \to X$, where $\Delta[n]$ denotes the standard n-simplex viewed as topological subspace of $\mathbb{R}^{n+1}$. (In particular, you need to specify what this functor does on morphisms!)

(2) Show that the monoidal category $\langle \mathrm{Rng}, \otimes_{\mathbb{Z}}, \mathbb{Z}\rangle$ is *not* closed.

(3) Construct a left adjoint for $\mathrm{Set}_*(S, -)\colon \mathrm{Set}_* \to \mathrm{Set}_*$, where Set_* is the category of sets together with a choice of element, with morphisms those set maps that take the basepoint of the source to the basepoint of the target.

(4) Show that, for $\langle \mathcal{B}, \otimes, \dots\rangle$ a closed monoidal category, there is an isomorphism $X^{Z\otimes Y} \simeq (X^Y)^Z$. (Hint: this follows from properties of adjunctions, without making use of the monoidal structure.)

(5) (CGHaus is closed. Long and somewhat in-depth, definitely challenging if you don't know point-set topology.) A topological space X is *compactly generated* when $C \subset X$ is closed in X if and only $C \cap K$ is closed in K for all $K \subset X$ compact (so compact subsets "detect" closed subsets).

Compactly generated Hausdorff spaces form a full subcategory of Haus, which we'll call CGHaus. In this exercise, you will work through the essential ingredients in the proof that the category CGHaus is closed, when given an appropriate monoidal structure.

(a) For a given Hausdorff space X, define a space KX as follows:
- The underlying set of X is the underlying set of KX.
- The closed sets of KX are those subsets $A \subset X$ so that for all $K \subset X$ compact, $A \cap K$ is closed.

Show that KX is a topological space, and that the natural map (which is the identity on underlying sets) $KX \to X$ is continuous. Verify that KX is compactly generated and Hausdorff. Show that $X \mapsto KX$ defines a right adjoint to the inclusion CGHaus $\hookrightarrow$ Haus, and explicitly write down a universal property of $KX \to X$ given by the counit.

(b) Show that, for X and Y in CGHaus, the space $K(X \times Y)$[8] is a product in the category CGHaus (hint: RAPL). Conclude that CGHaus has a natural monoidal structure.

(c) For X, Y compactly generated Hausdorff, define $\mathrm{Cop}(Y, X)$ to be the topological space with underlying set $\mathrm{Set}(Y, X)$, which is

[8] where $X \times Y$ is the product in Top, which coincides with the product in Haus

given a topology with subbasis consisting of sets $N(C,U)$ as C ranges over compact subsets of Y and U over open subsets of X. $N(U,C)$ contains a function $f\colon Y \to X$ if and only if $f(C) \subset U$. Define X^Y to be $K(\mathrm{Cop}(Y,X))$. Show that $(-)^Y\colon X \mapsto X^Y$ is a right adjoint to $- \otimes Y\colon Z \mapsto K(Z,Y)$ as follows:

(i) Show that the evaluation map $e\colon X^Y \times Y \to X$ given by $(f,y) \mapsto f(y)$ is continuous on sets of the form $D \times C$, where D is compact in $\mathrm{Cop}(Y,X)$ and C is compact in Y. Conclude that e is continuous on compact subsets, and hence continuous (using that all spaces involved are compactly generated).

(ii) Show that, given a continuous map $g\colon Z \otimes Y \to X$, there is a continous map $k\colon Z \to X^Y$ with $e \circ k \otimes 1 = g$.

(iii) Use the fact that, on in the category of sets, $\mathrm{Set}(Z \times Y, X) \simeq \mathrm{Set}(Z, \mathrm{Set}(Y,X))$ to conclude that this establishes a adjunction $\mathrm{CGHaus}(Z \otimes Y, X) \simeq \mathrm{CGHaus}(Z, X^Y)$ (use that the underlying set of X^Y is that of $\mathrm{Set}(Y,X)$, and that of $Z \otimes Y$ is $Z \times Y$ to conclude that k from the previous problem is uniquely determined, and that $g \mapsto k$ established a natural bijection).

(6) Show for Y Hausdorff, the compactly generated Hausdorff space KY defined in the previously problem is precisely the colimit in Haus of all compact subspaces of Y, ordered by inclusion.

(7) Show that CGHaus $\hookrightarrow$ Haus creates colimits.

(8) Show that the category CGHaus is equivalent to the category which has as objects all Hausdorff spaces (not necessarily compactly generated), and as morphisms all set maps $f\colon X \to Y$ (not necessarily continuous on all of X) which are continuous when restricted to compact subsets of X. (Define a functor and show it is fully faithful and essentially surjective on objects.)

(9) Show that the smash product $\wedge\colon \mathrm{CGHaus}_* \times \mathrm{CGHaus}_* \to \mathrm{CGHaus}_*$ is commutative and associative up to natural isomorphism, making CGHaus_* a symmetric monoidal category. What is the monoidal unit?

§17. Lecture 7/30/2018

§17.1. **Summary.**

- Since this is a short lecture, the idea is to go over some facts about limits and colimits that are pretty useful in various practical (mathematical) settings.
- Filtered categories. Filtered (co)limits (IX.1).
- Commuting limits and colimits (IX.2).
- Final functors for computing colimits (IX.3).

§17.2. **Exercises.**

(1) Precisely formulate the dual statement to the assertion that filtered colimits commute with finite limits. Think about how to "dualize the proof."

(2) Show that (finite) coproducts commute with pullbacks in Set . (This is in contrast to the fact, stated in class, the coproducts and products in set do not in general commute.) Dually, show that (finite) products commute with pushouts in Set .
(3) (Challenging.) Show that the forgetful functor Grp $\to$ Set creates filtered colimits. [9]
(4) (Final functors.) A functor $L\colon J' \to J$ is *final* if, for every element $k \in J$, the comma category $(k \downarrow L)$ is non-empty and *connected*, meaning that the underlying (undirected) graph of the comma category is nonempty and connected in the sense of graph theory (having exactly one component). That is, there is a way to "get from any object to the other," if you don't care about direction of arrows. Final functors are useful because of the following

    17.1. THEOREM. *If* $L\colon J \to J'$ *is final and* $F\colon J \to X$ *is a functor such that* $\operatorname{colim} FL$ *exists, then so does* $\operatorname{colim} F$ *and moreover* $\operatorname{colim} F = \operatorname{colim} FL$.

    In this exercise, you will get a sense for what it means to be a final functor, and you'll see why this result is useful.
    (a) Show by hand (i.e. using the structure of colimits in Set) that, in the category of sets, a colimit indexed by $\mathbb{N}$ is equal to a colimit calculated over any infinite subset.
    (b) Show that the inclusion of any infinite subset into $\mathbb{N}$ is final. Using the theorem, this gives an alternative proof of the previous part.
    (c) Prove that the composite of two final functors is final.
    (d) Suppose that $L\colon J' \to J$, and that for every $F\colon J \to \mathcal{C}$ the natural functor $\operatorname{colim} FL \to \operatorname{colim} F$ is an isomorphism. Show that L must be a final functor. (Hint: take F to be a representable functor and use the hypothesis.)
    (e) Let $j \in J$ and write $\{j\}$ for the discrete subcategory of J consisting of the element j. Show that $\{j\} \to J$ is final if and only if j is a terminal object in J. What does this tell you about computing the colimit of a diagram indexed by a category with a terminal object? Also dualize this statement to obtain an analogous result for limits ("final" will no longer be the correct condition).

[9] This is Proposition 2 in MacLane IX.1 if you get stuck, but I'd suggest you think about it on your own a bit first – it will require you to understand coproducts/ colimits in Grp (which are different from those in Ab) and work explicitly with the definition of creation of limits which I get the sense is (understandably) still a bit murky.

§18. Lecture 8/1/2018

§18.1. **Summary.**

- Symmetry in monoidal categories.
- Strictness and coherence; braidings in monoidal categories.
- The Artin braid group B_n and braid category B.
- Braided monoidal categories and B. "Representability result:" recovering the underlying category using B; implies coherence "up to braids" in braided monoidal categories.
- A bit about knot invariants and braided monoidal categories. (Reference: Kassel, "Quantum Groups." Available through Hollis.)

§18.2. **Exercises.**

(1) Convince yourself that the relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| > 1$ hold in B_n. Show that there are no other relations. (Level of rigor up to you.)

(2) (Not really related to the class.) Try to come up with a way of uniquely representing elements in B_n that makes it easy to distinguish elements (e.g. something similar to cycle notation for S_n, or a statement that there is a "canonical" way to write every element as a product of σ_i's). Disclaimer: I don't know the answer to this/ if there's even a nice answer! If it ends up being very difficult you might do some googling. Just looking up "braid group presentations" or something like this might be a good place to start. Let me know if you come up with a nice answer!

(3) Carefully work out the correspondence between $\pi_1(T_n)$ and B_n as defined geomtrically in terms of "concatenation of braids." (Where T_n is the space of n distinct, unordered points in $\mathbb{R}^n$, and B_n is the Artin braid group of braids on n strands, both as defined in class/ Mac Lane chapter XI.)

(4) Show that the braid category B is strict monoidal. Show that B is braided but not symmetric by verifying that the braiding defined in the e-mail after class is natural, but is not symmetric. Write down an algebraic formula for the braiding in terms of σ_i's.

(5) Show that, if $\langle M, \otimes, e, \gamma \rangle$ is a symmetric monoidal category, then commutative of either of the two hexagon diagrams for a braiding γ implies commutativity of the other.

(6) Show that the category of strong braided monoidal functors between two strict monoidal categories with braidings is equivalent to the category of *strict* braided monoidal functors, via the obvious inclusion of strict functors into strong functors.