Problem 10

The aim here is to prove the folowing:

Theorem. *Let G be a non-abelian p-group in which every subgroup is normal. Then $p = 2$.*

We make some observations on what must take place in G. The basic one is:

(1) Every element of order p in G is central.

Proof: Let c be an element of order p, and x any element of G. Then $\langle x\rangle \cap \langle c\rangle$ is either $\langle c\rangle$ or $\{e\}$. If the intersection of these two normal subgroups is trivial, then their elements pairwise commute by a previous exercise. In either case, x and c commute.

Choose some element y in G which is not central. We may assume that it has minimal order p^u among all non-central elements. By (1), $u > 1$. There is some element x in G that does not commute with y. Say its order is p^v. Let

$$c = y^{-1}x^{-1}yx \tag{2}$$

From the fact that $\langle x\rangle$ and $\langle y\rangle$ are normal, we easily deduce that c is in $\langle x\rangle \cap \langle y\rangle$. But every subgroup of $\langle x\rangle$ contains $x^{p^{v-1}}$ and every subgroup of $\langle y\rangle$ contains $y^{p^{u-1}}$. It follows that each of these elements is a power of c. As cyclic p groups contain a unique subgroup of order p, we must have

$$x^{ap^{v-1}} = y^{p^{u-1}}$$

for some integer a relatively prime to p. In fact we can suppose that $a = 1$; there would have been no problem replacing x by x^a. From (2) we get

$$x^{-1}yx = yc$$

It follows that:

$$x^{-1}y^px = (yc)^p = y^pc^p \tag{3}$$

By the minimality of the choice of y and u, y^p is central. It follows immediately from 3) that $c^p = 1$. By (1), we conclude that c is central in G. Thus we have a commutation rule:

$$yx = xyc$$

where c is a central element of order p. It follows that

$$yx^i = x^iyc^i$$

for all integers i. We are interested in powers of yx^i. A test case is:

$$(yx^i)^2 = x^i y c^i x^i y c^i = x^i y x^i y c^{2i} = x^i x^i y c^i y c^i = x^{2i} y^2 c^{3i}$$

The end result should be to prove by induction on t that:

$$(yx^i)^t = x^{it} y^t c^{i \cdot \frac{1}{2} t(t+1)} \tag{4}$$

We pick the following values for t and i in (4): set $t = p^{u-1}$ and $i = -p^{v-u}$. Now assume that p is odd. Since $u > 1$, the exponent of c is divisible by p (in fact by t), and the right hand side of (4) becomes trivial.

This is a contradiction; since yx^i does not commute with x, it is not a central element, but its order is a divisor of p^{u-1}, contradicting the minimality of u.