Field Theory Exercises

There is a sense in which Galois theory is just group theory warmed over. In any case, much of field theory is a consequence of group theory.

1) Suppose $F \subset K$ are fields with K/F separable. Let $\tilde{K}$ be the *Galois closure of* K/F, i.e. if $K = F(\alpha_1, a_2, \ldots, a_s)$ with f_i the irreducible polynomial of α_i over F, then $\tilde{K}$ is the splitting field of $f = f_1 f_2 \cdots f_s$ over F. Show this notion is well-defined independent of choices of the α_i or f_i.

2) Suppose $F \subset K \subset M$ with M/F Galois. Show that $\tilde{K}$ is (up to an isomorphism fixing F) the smallest Galois subfield of M containing K.

3) In the context of 2), show that $\tilde{K} = \sigma_1(K)\sigma_2(K)\cdots\sigma_r(K)$ where the $\sigma_i(K)$ represent the distinct images of K by elements of $G = G(M/F)$.

4) Take the setup in 2). Let H be the subgroup of $G = G(M/F)$ corresponding K in the Galois correspondence. Show that $\tilde{K}$ corresponds to

$$\bigcap_{\sigma \in G} \sigma H \sigma^{-1}$$

Show in particular, M is the Galois closure of K if and only if this intersection is trivial, and this happens if and only if H contains no nontrivial normal subgroup of G.

5) Show that the stabilizer of a point in S_n is a maximal subgroup.

6) Let M/F be a Galois extension with $G(M/F) \cong S_n$. Show that M is the splitting field over F of a monic irreducible polynomial of degree n. If u is a root of f in M, then $[F(u) : F] = n$, but there are no intermediate fields between F and $F(u)$ (whether or not n is composite).

7) Let M/F be Galois with Galois group S_5. Assume that all intermediate fields have a primitive element. Show that M is the splitting field over F of irreducible polynomials of degree 5,6,15,or 20, but not the splitting field of any irred. polynomial of degree 4 or 11.

8) Let M/F be Galois with Galois group dihedral of order 8. Assume every intermediate field has a primitive element. Show that M is the splitting field over F of an irred. polynomial of degree 4.

9) Let M/F be Galois with Galois group quaternion of order 8. Show that M cannot be the splitting field of any polynomial in $F[t]$ of degree n, $n < 8$.

10) Suppose M is Galois over F and K/F is a separable extension. Show that MK/K is Galois.

11) Suppose L and M are Galois extensions of F and $L \cap M = F$. Show that LM is Galois over F and $G(LM/F) \cong G(L/F) \times G(M/F)$ as a direct product.