Math 95: Transition to Upper Division Mathematics, Part II

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Contents

1 Introduction •

1.1 Office hours / lecture?

In this part of the class, your job is to:

1. read the notes that I have provided;
2. write proofs to the questions that I have provided;
3. except for some turned in problems, the only way you will know your proofs are correct is by checking them with me or Kevin;
4. "lecture" will be replaced with a time for you to come in, have me read through your proofs, so that I can correct any errors you have made; this will occur in the usual classroom; other office hours will be held as usual in my office;
5. if you have not read the notes needed to address a certain question, my comments concerning the question will be a waste of both of our time; I am happy to help clear up confusing aspects of the notes; I cannot read them for you.

1.2 Updates

I'll be updating these notes to try and make them better.

I'll put a • next to a section title when I think that section is unlikely to change any more.

1.3 Participation

Coming to class to check the correctness of your work will be rewarded with participation credit.

1.4 Homework

- Turn in 3 problems of your choice which you think you have done perfectly. Write the question out, so that Kevin does not have to look around for the question. Kevin will grade these for correctness.

  This is mainly so you can check you're doing the proofs correctly. If you're having your proofs checked in class, then you will already know this. Turning them in will allow you to get credit for them. In particular, you are allowed to turn in ones we have already graded as correct if you just want the points!

- Turn in three problems where you got stuck.
  - Highlight how the proof should look, i.e. give the template of the proof.
  - Highlight the gaps you cannot fill in.
  - Try and explain why you are stuck as clearly as possible.

1.5 My approach to teaching you quantifiers

I was about 5 or 6 years old when I learned to ride a bicycle. Of course, I started with stabilizers. (I think that in the US these are more frequently called training wheels.) When I had gotten used to my stabilizers, it was time for them to come off. I was taken to a sports field, and my seat was held firmly by a strong man (Graham was his name) as I tried pedalling without them for the first time. With his support, I did not fall off and hurt myself. Eventually, when I was ready, he ran along side me as I gathered speed, gave me a big push, and for the first time, I could ride a bike! I tried a few more times with the occasional fall. On the way home, I used my stabilizers again, because I did not want to fall off onto the pavement (sidewalk) which would be much more painful than the grass of the sports field.

When I learned to drive a car in preparation for my driving test, the instructor gave me a tip for my parallel parking. He told to look in my wing mirror; when I could see that the curb was in line with the top of my back tire, it was time to start turning the steering wheel the other way. I don't drive much in LA, but when I see people parallel park here, they often have to do something cleverer than that because parking is so difficult!

Both of these silly stories relate to the way that I want to teach you to write proofs. Most people don't use stabilizers when they ride a bicycle. But, I hope that you'll all agree that they are very useful when learning. Similarly, some of the techniques I will give you for writing proofs, namely, my quantifier verification procedure, are not used as explicitly by professional mathematicians. However, I have found that students find the procedure useful: for one thing, the procedure guides the structure of your proof, giving you a place to start thinking. As we write more proofs, in simple cases I might suggest you try to write a proof more like a professional mathematician would, and stray a little from the procedure. If, however, things get tricky, just as I used my stabilizers on my way home, you can always go back to the verification procedure. It will prevent you falling on the pavement, i.e. making a mistake!

Finally, mathematics does become difficult, just like parking in LA. Whatever I teach you about writing proofs, eventually one will show up where you have to do something cleverer than what you have done before. What can prepare you for this? Only experience, a full understanding of what you have learned up to that point, and the confidence gained from this. Good luck, mathematicians!

2 Truth Tables •

Mathematics is concerned with statements which are true or false. Here are some examples of such statements.

- $0 = 0$. True.
- 7 is an even number. False.
- 11 is a prime number. True.
- Every even integer greater than 2 can be written as the sum of two primes.

  Unknown, but definitely true or false, and not both.
- There are infinitely many prime numbers p with the property that $p + 2$ is also prime.

  Unknown, but definitely true or false, and not both.

Just like when we speak English or any other language, mathematical statements are built up by putting together lots of simpler ones. In this section, we learn a few simple ways of constructing statements which are true or false from existing statements which are true or false.

2.1 NOT P

Suppose that P is a statement which is true or false (but not both). We normally call such a thing a *proposition*. We have a new proposition NOT(P), called the *negation* of P. The truth of NOT(P) is completely determined by the truth of P. This is demonstrated in the following truth table.

P	NOT(P)
T	F
F	T

We illustrate further using the first three examples from above.

- $0 \neq 0$. False.
- 7 is an NOT an even number. True.
- 11 is NOT a prime number. False.

It is more complicated to apply NOT to the last two examples from above but you'll be able to do this later.

You can see that NOT(NOT(P)) always has the same truth value as P. "I am NOT NOT your teacher for math 95" is true.

2.2 P and Q, P or Q

The truth tables for (P and Q) and (P or Q) are as follows.

P	Q	P and Q	P or Q
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

- $(0 = 0)$ and (11 is a prime number). True.
- $(0 = 0)$ or (11 is a prime number). True.
- (11 is a prime number) and (7 is an even number). False.
- (11 is a prime number) or (7 is an even number). True.
- (7 is an even number) and $(0 \neq 0)$. False.
- (7 is an even number) or $(0 \neq 0)$. False.

The second statement is true! It is important to note that, mathematically, "or" is taken to be inclusive unless we explicitly say otherwise.

I admit that the sentences above involving "or" feel awkward. Soon enough, we'll see sentences involving "or" which feel more natural.

2.3 $P \implies Q$

Here is the truth table for $(P \implies Q)$.

P	Q	$P \implies Q$	NOT$(P \implies Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

The following examples may look kind of ridiculous to you.

- $(0 = 0) \implies$ (11 is a prime number). True.
- (11 is a prime number) $\implies$ (7 is an even number). False.
- (7 is an even number) $\implies$ $(0 = 0)$. True.
- (7 is an even number) $\implies$ $(0 \neq 0)$. True.

Remark 2.3.1. Something strange but sensible is occurring with $P \implies Q$. We'll remark on the strange aspects here; we'll be able to talk about the sensible part once we have propositional *functions* and universal quantifiers.

1. Many people say $P \implies Q$ as "if P, then Q." This makes it sound like the truth of P could change. However, if P is a proposition, it has a fixed truth value, so does Q, and so does $P \implies Q$. Once we have propositional *functions*, for which truth values can change, the languange "if P, then Q" will seem more reasonable.
2. Many people say $P \implies Q$ as "P implies Q," and this suggests some degree of causation. Mathematically, the truth of $P \implies Q$ is observational. Even once we introduce propositional functions, $P \implies Q$ statements can be true for no good reason. Is 11 prime because $0 = 0$?!
3. At this moment in time, I would prefer for you to say "P arrow Q," and I will try my best to do so as well. This emphasizes the more formal meaning of $P \implies Q$, i.e. the truth table.
4. Finally, notice that $P \implies Q$ is taken to be true when P is false. This will make sense once we talk about propositional functions, universal quantifiers, and "if-then" sentences.
5. Final finally (that you might ignore!), there are examples where "if-then" can sound more reasonable for propositions. For instance, "if the Riemann Hypothesis is true, then the twin prime conjecture is true." The reason is that we don't know about the truth of the individual clauses, but can still make arguments under the assumption the Riemann Hypothesis is true.

2.4 $P \iff Q$

$P \iff Q$ is shorthand for $((P \implies Q)$ and $(Q \implies P))$.

P	Q	$P \implies Q$	$Q \implies P$	$P \iff Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$P \iff Q$ is true exactly when P and Q have the same truth values.

2.5 De Morgan's Laws, NOT($P \implies Q$)

De Morgan's Laws state the following propositions are true f or absolutely any propositions P and Q (such propositions are called *tautologies*).

1. NOT(P or Q) $\iff$ (NOT(P) and NOT(Q));
2. NOT(P and Q) $\iff$ (NOT(P) or NOT(Q)).

Thus, we may as well regard the propositions on the left of the $\iff$ s as the same as the respective ones on the right of the $\iff$ s, and use "is" instead of $\iff$:

1. NOT(P or Q) is (NOT(P) and NOT(Q));
2. NOT(P and Q) is (NOT(P) or NOT(Q)).

These rules allows us to *negate* a longer sentence by negating the shorter parts of it and swapping "and" with "or."

- You can use a truth table to see that

$$(P \implies Q) \text{ is } (\text{NOT}(P) \text{ or } Q).$$

- De Morgan's Laws tell us that

$$\text{NOT}(P \implies Q) \text{ is } (P \text{ and NOT}(Q)).$$

2.6 Exercises

These exercises are left over from a previous class. While you are welcome to try them, they are probably quite easy, quite boring, and I'm not sure that checking the truth tables is that useful.

Thinking about what they mean for a little bit would be more helpful.

1. By using truth tables, demonstrate that De Morgan's laws are true.
   - NOT(P or Q) $\iff$ (NOT(P) and NOT(Q));
   - NOT(P and Q) $\iff$ (NOT(P) or NOT(Q)).
2. Verify that ($P \implies Q$) $\iff$ (NOT(P) or Q) is always true.
3. Demonstrate that the following propositions are always true.
   - (Cases) ((P or Q) and NOT(P)) $\implies Q$.
   - (Modus ponens) (P and ($P \implies Q$)) $\implies Q$.
   - (Contrapositive) ($P \implies Q$) $\iff$ (NOT(Q) $\implies$ NOT(P)).
   - (Contradiction) $P \iff$ (NOT(P) $\implies 0 \neq 0$).

   Deduce from the last proposition that the following proposition is always true.

$$(Q \implies R) \iff ((Q \text{ and NOT}(R)) \implies 0 \neq 0).$$

3 Universal quantifiers and "if-then" sentences •

So far the sentences that we have considered have not been very interesting. This is because each of the following sentences only makes reference to one number at a time.

- $0 = 0$.
- 7 is an even number.
- 11 is a prime number.

More interesting mathematical statements include variables. Variables allow us to talk about many numbers at the same time. For instance, we might make a statement about every real number x.

Also, $P \implies Q$ still feels like a very strange thing, almost unrelated to how you have (mis)used the symbol $\implies$ up to now. The purpose of this section is to be able to express simple "if-then" sentences correctly symbolically. For example, we'll see that "if x is a real number bigger than 10, then x is bigger than or equal to -10" is expressed symbolically as

$$\forall x \in \mathbb{R},\ x > 10 \implies x \geq -10.$$

There are two issues we have to address. First, without careful analysis, it looks like the truth of

$$x > 10 \implies x \geq -10$$

could possibly depend on x, that is, it is not a proposition, but a *propositional function*. Secondly, we need to talk about what "$\forall x \in \mathbb{R}$" means.

3.1 Propositional functions

A propositional function is a statement whose truth has the possibility of depending on variables. As an example, let's wonder whether the following equation is true.

$$x = 0$$

It depends on what x is. If x is 0, then the equation is true, but if x is 1 then the equation is false. So rather than having one truth value, there is a truth function depending on x. The function gives "true" if $x = 0$, and "false" if $x \neq 0$.

Remark 3.1.1. Another problem, which we won't pay much attention to right now, is that we're not entirely sure what type of mathematical object x is supposed to be in the above equation. Is it a natural number, an integer, a rational number, a real number, a complex number, a quarternion, a vector, a matrix, a vector space, an element of a finite field…? The list is endless: zero is a very common concept in mathematics so the right hand side of the equation does not help us much! I meant it to be a real number, and I hoped that this is how you would interpret it.

All of our ways of constructing new propositions out of old ones apply to propositional functions. For instance, all of the following are propositional functions.

Example 3.1.2.

- NOT$(x = 0)$. This is the same as $(x \neq 0)$.
- $(x > 0)$ and $(x^2 < 1)$.
- $(x = 0)$ or $(x > 0)$. This is often abbreviated as $(x \geq 0)$.
- $(x > -10) \implies (x \geq 10)$.

The corresponding truth functions are respectively as follows.

- True when $x \neq 0$; false when $x = 0$.
- True when $0 < x < 1$; false otherwise.
- True when $x \geq 0$; false when $x < 0$.
- False when $-10 < x < 10$; true otherwise.

The last is the one which requires the most thought. I would suggest going through the relevant truth table separately in the cases when $x \leq -10$, $-10 < x < 10$, and $x \geq 10$.

3.2 Universal quantifiers $\forall$

English is a very assertive language. If I say "cats are grey," then I really mean "*all* cats are grey," and of course, I am incorrect because you have seen a ginger cat. (Sidenote: "All Cats Are Grey" is a song by The Cure, a pretty solid effort.)

Similarly, if I say "boys are stupid," I am asserting, that not one, not two, but that every single boy in the world is stupid. A bizarre way of saying what I meant is, "for all boys b, b is stupid."

"For all" is called the *universal quantifier*, and the mathematical symbol for it is $\forall$. Suppose $P(x)$ is a propositional function, where x ranges through real numbers. The sentence

$$\forall x \in \mathbb{R},\ P(x)$$

is read as "for all real numbers x, $P(x)$." It is true exactly when $P(x)$ is true for all real numbers x. If there is at least one real number x for which $P(x)$ is false, then the statement is false. Let's see some examples.

- $\forall x \in \mathbb{R},\ x = 0$. This is false because 1 is a real number and $1 \neq 0$.
- $\forall x \in \mathbb{R},\ x^2 > 0$. This is false because 0 is a real number, $0^2 = 0$, and 0 is not *strictly* bigger than 0.
- $\forall x \in \mathbb{R},\ x^2 \geq 0$. This is true.
- $\forall x \in \mathbb{R},\ (x+1)^2 = x^2 + 2x + 1$. This is true.

In each of these, cases, by using $\forall$, we have gone from a propositional function to a proposition, a sentence with whose truth value has no possibility for variable dependence.

If we have a propositional function of two variables x and y, we need two quantifiers to return to a proposition. For example, the following sentence is true.

$$\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x^2 + y^2 \geq 0.$$

3.3 $P \implies Q$ revisited with propositional functions and universal quantifiers

Consider the following two sentences.

$$\forall x \in \mathbb{R},\ (x > -10 \implies x \geq 10).$$

$$\forall x \in \mathbb{R},\ (x > 10 \implies x \geq -10).$$

We noted in example 3.1.2 that $(x > -10 \implies x \geq 10)$ is false when $x = 0$. This is given by the truth table for " $\implies$ " because $(0 > -10)$ is true and $(0 \geq 10)$ is false. For this reason, the first sentence is false.

On the other hand, the second sentence is true. To see this we must check that for every real number x, $(x > 10 \implies x \geq -10)$ is true. We can do this by considering two cases. Let $x \in \mathbb{R}$.

1. Case 1: $x \leq 10$.

   In this case, $(x > 10)$ is false, so the truth table for " $\implies$ " says $(x > 10 \implies x \geq -10)$ is true.

2. Case 2: $x > 10$.

   In this case, $(x > 10)$ is true. To check that $(x > 10 \implies x \geq -10)$ is true we must verify that $(x \geq -10)$ is true. It is true because $x > 10 \geq -10$.

3.4 "If-then" sentences

So far in this section we have discussed the following concepts.

- Propositional functions.
- Universal quantifiers.
- $P \implies Q$ again.

These concepts combine to allow us to express many "if-then" sentences in math symbolically. To see this, suppose that $P(x)$ and $Q(x)$ are propositional functions of one variable, a real number. What does it mean for the following sentence to be true?

$$\forall x \in \mathbb{R},\ P(x) \implies Q(x).$$

This means that for all $x \in \mathbb{R}$, $(P(x) \implies Q(x))$ is true. So for each real number x,

- either $P(x)$ is false;
- or $P(x)$ is true and $Q(x)$ is true.

We can express this in English by saying, "if $P(x)$ is true, then $Q(x)$ is true." Normally, in English, we also miss out the "is true" and say "if $P(x)$, then $Q(x)$."

The symbolic sentences of the last subsection express the following "English sentences."

- Suppose x is a real number. If x is bigger than -10, then x is bigger than or equal to 10.
- Suppose x is a real number. If x is bigger than 10, then x is bigger than or equal to -10.

In the first case, you'd say, "No, that's not true because 0 is bigger than -10, but it is less than 10." In the second case, you would say, "yes, that's true because $10 \geq -10$." This is the reasoning that we gave when we thought about things more formally.

I hope this discussion illustrates why the truth table for $P \implies Q$ is chosen the way it is. You might need to sit and think about this for an hour or so!

Notice that math takes an innocent-until-proven-guilty perspective to consequential statements. Suppose that we lived in an alternative universe where Prince (the one who made the album "Purple Rain") said,

"at all moments of my life, if it rains, then I carry an umbrella"

and for his entire life, it never rained. Regardless of whether he did or did not carry an umbrella at various points of his life, you could not prove that he lied to you. So, mathematically, he told the truth even if, secretly, he planned for the heavens to open during his Superbowl performance just to make that rendition of "Purple Rain" in 6-inch heels with a purple Love-Symbol-shaped guitar even more epic. (He didn't have an umbrella when this happened and his guitar was waterlogged.)

This has some strange implications. For example, the following symbolic sentence is true.

$$\forall x \in \mathbb{R},\ x \neq x \implies 0 = 1.$$

It says, "suppose x is a real number; if $x \neq x$, then $0 = 1$." This may seem a little unusual, but if you read it sarcastically enough, you can make it sound okay. The following symbolic sentence is also true.

$$\forall x \in \mathbb{R},\ x \neq x \implies x = 0.$$

It says, "suppose x is a real number; if $x \neq x$, then $x = 0$." Even sarcasm can't make this sentence sound reasonable because $x = 0$ is not obviously false; in fact, it is true sometimes. Perhaps it is best to start stuttering after saying "if $x \neq x$," have an existential crisis, and conclude that nothing means anything, and that everything is temporarily true.

At the end of the day, one just has to accept that there are certain sentences which are true even though they sound strange. It's the same in English: "Donald Trump is president of this country." Thankfully, in math, these sentences are often useless. If you take a class in mathematical logic, you'll also find that there are gramatically correct mathematical sentences that are weird. This is like English too (see 1:50, and particularly 2:12, of `www.youtube.com/watch?v=MSyIhapMdI8`).

3.5 Implicit "for all"s

When we first introduced universal quantifiers, we remarked that in English we use many implicit universal quantifiers. The sentence "boys are stupid" really means "all boys are stupid" and "if it rains, then I carry an umbrella" really means "at all moments of my life, if it rains, then I carry an umbrella."

Implicit universal quantifiers are frequently used in mathematics too and we will see important examples later. This is unfortunate since such omissions can create confusion. I encourage you to be explicit about all universal quantifiers. Sometimes this will be essential in order to be correct. The nice thing about writing sentences symbolically, is that omitting quantifiers is incorrect. So it forces us to identity such universal quantifiers.

3.6 Another example and proof by cases

In any math class, it is important to make clear to the students what results can be assumed...

A *construction* of the rational numbers and real numbers together with the ordering given by $<$ is possible. From such a construction, it is possible to prove all of the standard properties of $\mathbb{Q}$, $\mathbb{R}$, and $<$ that you should be used to from high school mathematics. Such a story would be quite confusing right now. For this reason, I'll just assume such standard properties. So, for now, there are some facts that we just have to believe. In fact, one never gets away from this in math. Math is more like a religion than you might think: mathematicians have to believe in the axioms of set theory; proving their consistency would mean that they are inconsistent (Gödel)!!

There is one property of the reals that students are frequently not careful enough with. Here it is.

Theorem 3.6.1. *(A property of $<$) The following is true.*

$$\forall a \in \mathbb{R},\ \forall b \in \mathbb{R},\ \forall c \in \mathbb{R},\ (a > 0 \text{ and } b > c) \implies ab > ac.$$

The issue is that the following sentence is NOT true because $a = -1$ is problematic: multiplication by -1 reverses inequalities.

$$\forall a \in \mathbb{R},\ \forall b \in \mathbb{R},\ \forall c \in \mathbb{R},\ (b > c) \implies ab > ac.$$

I would like to give another example of a universally quantified sentence. I would also like to introduce proofs that make use of cases, and proofs which reference a previous result. The following result serves that purpose. However, as mentioned at the beginning of this subsection, you can assume its truth in other contexts because it is a standard result about $\mathbb{R}$ and $<$.

Theorem 3.6.2. *(A related property of $<$) The following is true.*

$$\forall a \in \mathbb{R},\ \forall b \in \mathbb{R},\ \forall c \in \mathbb{R},\ (a \geq 0 \text{ and } b \geq c) \implies ab \geq ac.$$

Proof. First, we should declare our variables. I'll do this according to a number system which we'll use later. This number system has more of an advantage once existential quanitfiers are introduced.

1. Let $a \in \mathbb{R}$.
2. Let $b \in \mathbb{R}$.
3. Let $c \in \mathbb{R}$.
4. We must verify $(a \geq 0$ and $b \geq c) \implies ab \geq ac$. Looking at the truth table for $\implies$, we see that we can break into two cases.
   (a) $(a \geq 0$ and $b \geq c)$ is false. The arrowed statement is then true.
   (b) $(a \geq 0$ and $b \geq c)$ is true. We must check that $ab \geq ac$ is true.
   I want to use the previous theorem. The only problem is that we have $\geq$ showing up instead of $>$, so there are more cases to consider.

i. $a = 0$.
In this case, $ab = 0$ and $ac = 0$, so $ab = ac$. In particular, $ab \geq ac$.

ii. $b = c$. In this case $ab = ac$. In particular, $ab \geq ac$.

iii. $a > 0$ and $b > c$.
In this case, we can use the previous theorem to conclude that $ab > ac$. In particular, $ab \geq ac$.

$\square$

Remark 3.6.3. Notice that our cases can overlap. For instance, we could have $a = 0$ and $b = c$ at the same time. To me, this seems like some motivation for having "or" be inclusive in mathematics.

Remark 3.6.4. When proving this I naturally broke into the cases that I just wrote in the proof. But some experience was used in doing this. Clearer cases for you might have been the following.

- $a = 0$ and $b = c$.
- $a = 0$ and $b > c$.
- $a > 0$ and $b = c$.
- $a > 0$ and $b > c$.

Have a think about how your proof would look based on such a decision.

In the other direction, I could have condensed to two cases.

- $a = 0$ or $b = c$.
- $a > 0$ and $b > c$.

I found a middle ground between repeating myself a lot, and risking that my writing would become unclear.

Remark 3.6.5. You should note the way that I use "in particular." "In particular" is used in the following way.

[Something holding less often]. In particular, [something holding more often].

Some examples in plain English are as follows.

I am a man. In particular, I am a human.
I am a human. In particular, I am an animal.
I love all burgers. In particular, I love In 'N' Out burgers.

Some math examples.

$x = y$. In particular, $x \leq y$.
$x > y$. In particular, $x \neq y$.
$n \in \mathbb{Z}$ is divisible by 4. In particular, n is even.

3.7 More examples: absolute value (it is safe just to skim this until you need the results)

Definition 3.7.1. The absolute value function $|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}$ is defined piecewise by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Lemma 3.7.2. $\forall x \in \mathbb{R},\ |x| \geq 0.$

Proof.

1. Let $x \in \mathbb{R}$.
2. We break into two cases.
   (a) $x \geq 0$.
   In this case, the definition of absolute value gives $|x| = x$. Since $x \geq 0$, this gives $|x| \geq 0$.
   (b) $x < 0$.
   In this case, the definition of absolute value gives $|x| = -x$ and we have $-x > 0$. Thus, $|x| > 0$. In particular, $|x| \geq 0$.

□

Lemma 3.7.3. $\forall x \in \mathbb{R},\ |x| = |-x|.$

Proof.

1. Let $x \in \mathbb{R}$.
2. We break into two cases.
   (a) $x = 0$.
   In this case $x = -x$, so $|x| = |-x|$.
   (b) $x > 0$.
   In this case, the definition of absolute value gives $|x| = x$.
   Also, in this case, $-x < 0$, so the definition of absolute value gives $|-x| = -(-x)$.
   We conclude that $|x| = x = -(-x) = |-x|$.
   (c) $x < 0$.
   $-x > 0$, so we can use the previous case to say $|-x| = |-(-x)|$, i.e. $|-x| = |x|$.

□

Lemma 3.7.4. $\forall x \in \mathbb{R},\ x \leq |x|.$

Proof.

1. Let $x \in \mathbb{R}$.
2. We break into two cases.

(a) $x \geq 0$.
In this case, $x = |x|$. In particular, $x \leq |x|$.

(b) $x < 0$.
Since $0 \leq |x|$, we obtain $x < |x|$. In particular, $x \leq |x|$.

□

Corollary 3.7.5. $\forall x \in \mathbb{R},\ -|x| \leq x \leq |x|$.

Proof.

1. Let $x \in \mathbb{R}$.
2. The previous lemma says $x \leq |x|$. Since $-x \in \mathbb{R}$ the previous lemma also says that $-x \leq |-x|$. The first lemma says that $|-x| = |x|$, so this gives $-x \leq |x|$ and thus, $-|x| \leq x$.

□

Lemma 3.7.6. $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ |x| \leq |y| \iff -|y| \leq x \leq |y|$.

Proof.

1. Let $x \in \mathbb{R}$.
2. Let $y \in \mathbb{R}$.
3. From the truth table for $P \iff Q$, we see that we must show that it is not possible for exactly one of $|x| \leq |y|$, $-|y| \leq x \leq |y|$ to be true. This time, we will do this by showing that whenever one of them is true, the other is also true.
   (a) Suppose $|x| \leq |y|$ is true. Then $-|y| \leq -|x|$ is true too. By using the previous corollary, we see that $-|y| \leq -|x| \leq x \leq |x| \leq |y|$ is true; in particular, $-|y| \leq x \leq |y|$ is true.
   (b) Suppose that $-|y| \leq x \leq |y|$ is true. Then $x \leq |y|$ and $-x \leq |y|$ are true.
      i. Case 1: $x \geq 0$. We see that $|x| = x \leq |y|$ is true.
      ii. Case 2: $x < 0$. We see that $|x| = -x \leq |y|$ is true.

□

Theorem 3.7.7. *(The triangle inequality)* $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ |x+y| \leq |x| + |y|$.

Proof.

1. Let $x \in \mathbb{R}$.
2. Let $y \in \mathbb{R}$.
3.
   - By the previous lemma we have $-|x| \leq x \leq |x|$.
   - By the previous lemma we have $-|y| \leq y \leq |y|$.
   - Because of how inequalities work this gives
   $$-|x| - |y| \leq x + y \leq |x| + |y|,$$
   i.e. $-(|x| + |y|) \leq x + y \leq |x| + |y|$. By the previous lemma, we have $|x+y| \leq |x| + |y|$.

□

3.8 Exercises

For each of the symbolic sentences below, say whether they are true or false.

You'll be able to prove your results after further reading.

1. $\forall x \in \mathbb{R},\ \left[x^2 > 0 \implies x > 0\right].$

2. $\forall x \in \mathbb{R},\ \left[x > 0 \implies x^2 > 0\right].$

3. $\forall x \in \mathbb{R},\ \left[(x > 2 \implies x^2 > 4) \text{ and } (x > 0)\right].$

4. $\forall x \in \mathbb{R},\ x > 2 \implies \left[(x^2 > 4) \text{ and } (x > 0)\right].$

5. $\forall x \in \mathbb{R},\ x > 0 \implies \left[\forall y \in \mathbb{R},\ x + y > y\right].$

6. $\forall x \in \mathbb{R},\ x > 1 \implies \left[\forall y \in \mathbb{R},\ xy > y\right].$

7. $\forall x \in \mathbb{R},\ \left[(x > 0) \text{ or } (x > 1 \implies x = 0)\right].$

8. $\forall x \in \mathbb{R},\ \left[(x > 0) \text{ or } (x > 1)\right] \implies x = 0.$

Solutions:

1. False.

2. True.

3. False.

4. True.

5. True.

6. False.

7. True.

8. False.

4 Existential quantifiers; their relation to universal quantifiers •

4.1 Existential quantifiers

Even with all the concepts we have introduced so far, we are still unable to say a lot of very simple mathematical statements. Here's an example. Suppose I fix an integer n. What does it mean for n to be even?

Well, we should be able to divide n by 2 and get an integer k. If $\frac{n}{2} = k$, then we have $n = 2k$. To say n is even we are asserting that *there exists* an integer k such that $n = 2k$.

"There exists" is called the *existential quantifier*. The mathematical symbol for it is $\exists$. Suppose $P(x)$ is a propositional function, where x ranges through real numbers. The sentence

$$\exists x \in \mathbb{R} : \; P(x)$$

is read as "there exists a real number x such that $P(x)$." It is true exactly when there is at least real number x for which $P(x)$ is true. If $P(x)$ is false for all real numbers x, then the statement is false. Let's see some examples.

- $\exists x \in \mathbb{R} : \; x = 0$. This is true because 0 is a real number and $0 = 0$.
- $\exists x \in \mathbb{R} : \; x^2 > 0$. This is true because 1 is a real number, $1^2 = 1$, and $1 > 0$.
- $\exists x \in \mathbb{R}, \; x = x + 1$. This is false.

Moreover, we can complete the example that motivated introducing $\exists$.

Definition 4.1.1. Suppose $n \in \mathbb{Z}$. We say that n is *even* **iff** the following sentence concerning n is true.

$$\exists k \in \mathbb{Z} : \; n = 2k.$$

4.2 NOT, $\forall$, $\exists$

Let's reread what I wrote when I introduced $\forall$ and $\exists$.

- The sentence $(\forall x \in \mathbb{R}, \; P(x))$ is true exactly when $P(x)$ is true for all real numbers x. If there is at least one real number x for which $P(x)$ is false, then the statement is false.
- The sentence $(\exists x \in \mathbb{R} : \; P(x))$ is true exactly when there is at least real number x for which $P(x)$ is true. If $P(x)$ is false for all real numbers x, then the statement is false.

Reading this carefully allows you to see that

- $\mathrm{NOT}(\forall x \in \mathbb{R}, \; P(x)) = (\exists x \in \mathbb{R} : \; \mathrm{NOT}(P(x)))$.
- $\mathrm{NOT}(\exists x \in \mathbb{R} : \; P(x)) = (\forall x \in \mathbb{R}, \; \mathrm{NOT}(P(x)))$.

4.3 The negation of an "if-then" sentence

Recall that $(\forall x \in \mathbb{R},\ P(x) \implies Q(x))$ expresses "suppose x is a real number; if $P(x)$, then $Q(x)$." We are now able to negate this. We obtain

$$\exists x \in \mathbb{R} :\ P(x) \text{ and } \mathrm{NOT}(Q(x)).$$

This consolidates what we said earlier. The sentence

"at all moments of my life, if it rains, then I carry an umbrella"

is false as long as you can find at least one moment of my life when it is raining and I am NOT carrying an umbrella.

As a mathematical example, the negation of $(\forall x \in \mathbb{R},\ x^2 > 0 \implies x > 0)$ is

$$\exists x \in \mathbb{R} :\ x^2 > 0 \text{ and } x \leq 0.$$

4.4 Exercises

Negate the sentences in exercise 3.8 and justify to yourself that the truth value changes appropriately.

5 Quantifiers •

5.1 Universal and existential quantifiers

So far we have seen sentences which use only $\forall$, and sentences which use only $\exists$. Many interesting sentences can be constructed by using $\forall$ and $\exists$. Here are two examples.

$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R} :\ x < y.$$

$$\exists y \in \mathbb{R} :\ \forall x \in \mathbb{R},\ x < y.$$

What might seem a little strange is that one of these sentences is true, and one is false. The order of the quantifiers is very important.

It is surprisingly tricky to give the definition of truth of a quantified sentence. I think the best way to demonstrate what it means for a quantified sentence to be true is to give you a procedure for verifying the truth of a quantified sentence. It was not so clear how I expected you to justify your solutions to exercises 3.8. The next subsection gives a systematic way to answer such problems.

5.2 Verifying a simple sentence involving quantifiers

The simplest type of sentence we will encounter in this course will be of the form

$$Q_1 Q_2 \dots Q_n\ \varphi$$

where each individual Q_i is either of the form $\forall x_i \in X_i$ or $\exists x_i \in X_i$, and φ is something constructed from a load of formulae and the operations of "and," "or," and "$\implies$."

(We will need to think a bit more when φ starts involving quantifiers. In fact, we will need a whole new discussion if φ looks like $(\exists x :\ F_1) \implies F_2$, or $(\forall x,\ \exists y :\ F_1) \implies F_2$.)

- A proof of the truth of such a sentence should consist of $(n+1)$ steps.
- At the i-th step, to deal with $\forall x_i \in X_i$ you simply say "let $x_i \in X_i$."
- At the i-th step, to deal with $\exists x_i \in X_i$ you should say "let $x_i =$ BLAH" where BLAH needs to be an element of X_i written *only* in terms of variables which have already been specified. You should always check that the element x_i that you declare *is an element of the set* X_i.
- φ will be some propositional function involving the variables $x_1, \dots, x_n$. At the $(n+1)$-st step, you must verify the truth of φ based on how you declared the variables. For clarity, you should break this step up in to further substeps (a), (b), (c), ...

5.3 Examples

Example 5.3.1. The following sentence is true.

$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}:\ x < y.$$

Proof.

1. Let $x \in \mathbb{R}$.
2. Let $y = x + 1$.
3. Then $x < x + 1 = y$.

$\square$

Remark 5.3.2. There's something important to note about this proof: we let $y = x + 1$. This is fine because we made this declaration in step 2, and x was declared in step 2. Letting $x = y - 1$ in step 1 would have been an illegal declaration. We CANNOT define a variable in terms of a variable that's declared later.

Example 5.3.3. The following sentence is true.

$$\exists x \in \mathbb{R}:\ \forall y \in \mathbb{R},\ \exists z \in \mathbb{R}:\ x^2 + y^2 = z^2.$$

Proof.

1. Let $x = 0$.
2. Let $y \in \mathbb{R}$.
3. Let $z = y$.
4. We see that $x^2 + y^2 = 0^2 + z^2 = z^2$.

$\square$

Remark 5.3.4. There's something important to note about this proof: we let $z = y$. This is fine because we made this declaration in step 3, and y was declared in step 2. Letting $x = y$ in step 1 would have been an illegal declaration. We CANNOT define a variable in terms of a variable that's declared later.

Example 5.3.5. The following sentence is true.

$$\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \left(x \leq y \implies \left(\exists z \in \mathbb{R}:\ x + z^2 = y\right)\right).$$

Proof.

1. Let $x \in \mathbb{R}$.
2. Let $y \in \mathbb{R}$.
3. We must verify

$$x \leq y \implies \left(\exists z \in \mathbb{R}:\ x + z^2 = y\right).$$

   (a) Case 1: $x > y$.
   $x \leq y$ is false, so the truth table for "$\implies$" says the relevant sentence is true.

   (b) Case 2: $x \leq y$.
   $x \leq y$ is true and we must verify

$$\exists z \in \mathbb{R}:\ x + z^2 = y.$$

   Notice that because the original sentence is not of the nice form I mentioned at the start of the section, we've hit a point where we have to verify *another* proposition involving a quantifier.

   i. Let $z = \sqrt{y - x}$. z is a well-defined real number: we are working under the assumption that $x \leq y$, and so $y - x \geq 0$.
   ii. We must verify that $x + z^2 = y$. This is algebra:

$$x + z^2 = x + (\sqrt{y - x})^2 = x + (y - x) = y.$$

□

5.4 Proving a sentence involving quantifiers to be false

Example 5.4.1. Let's try to show the following sentence to be true.

$$\exists x \in \mathbb{R} : \ \forall y \in \mathbb{R}, \ \forall z \in \mathbb{R}, \ x^2 + y^2 = z^2.$$

Our proof would read as follows.

1. Let $x = \text{BLAH}$.
2. Let $y \in \mathbb{R}$.
3. Let $z \in \mathbb{R}$.
4. We need to check the truth of $x^2 + y^2 = z^2$.

We need to think up what BLAH should be. Suppose we tried $x = 0$ in step 1. Then, in step 4, we would have to show that $y^2 = z^2$. Checking that this is true is impossible since we know absolutely nothing about y and z (other than the fact that they are real numbers). y could be 1, and z could be 0, and in this case the equation is false. Similarly, if we tried $x = 1$ in step 1, then, in step 4, we would have to show that $1 + y^2 = z^2$. Checking that this is true is impossible too.

Hmm, it seems the sentence is false. How do we verify this? First, let's note some things.

- The reason it is false is not because *one* choice of x failed us; it is because *all* choices of x fail us. Saying it another way, for all $x \in \mathbb{R}$, the following proposition is false.

  $$\forall y \in \mathbb{R}, \ \forall z \in \mathbb{R}, \ x^2 + y^2 = z^2.$$

- Why is the proposition above false? When $x = 0$, we argued it was false by thinking about the case when $y = 1$ and $z = 0$. In the case when $x = 0$, this actually gives a proof that

  $$\exists y \in \mathbb{R} : \ \exists z \in \mathbb{R} : \ x^2 + y^2 \neq z^2$$

  is true.

- In the end, the reason that the orginal sentence is false is that the following sentence is true.

  $$\forall x \in \mathbb{R}, \ \exists y \in \mathbb{R} : \ \exists z \in \mathbb{R} : \ x^2 + y^2 \neq z^2.$$

Example 5.4.2. The following sentence is true.

$$\forall x \in \mathbb{R}, \ \exists y \in \mathbb{R} : \ \exists z \in \mathbb{R} : \ x^2 + y^2 \neq z^2.$$

Proof.

1. Let $x \in \mathbb{R}$.
2. Let $y = 1$.
3. Let $z = 0$.
4. We see that $x^2 + y^2 = x^2 + 1 \geq 1 > 0 = z^2$. In particular, $x^2 + y^2 \neq z^2$.

$\square$

We recap all we know about negation.

Definition 5.4.3. The *negation* of a sentence $\mathscr{S}$ is NOT($\mathscr{S}$); NOT($\mathscr{S}$) can be obtained from $\mathscr{S}$ via the following rules.

- Being careful with parentheses when there is a complicated arrangement of clauses;
- $\mathrm{NOT}(\exists x \in X : \ \varphi)$ is $(\forall x \in X, \mathrm{NOT}(\varphi))$;
- $\mathrm{NOT}(\forall x \in X, \ \varphi)$ is $(\exists x \in X : \mathrm{NOT}(\varphi))$;
- $\mathrm{NOT}(\mathrm{NOT}(\varphi))$ is φ;
- $\mathrm{NOT}(\varphi \text{ or } \psi)$ is $(\mathrm{NOT}(\varphi) \text{ and } \mathrm{NOT}(\psi))$;
- $\mathrm{NOT}(\varphi \text{ and } \psi)$ is $(\mathrm{NOT}(\varphi) \text{ or } \mathrm{NOT}(\psi))$;
- $\mathrm{NOT}(\varphi \implies \psi)$ is $(\varphi \text{ and } \mathrm{NOT}(\psi))$.

I'm not a fan of "rules." "Rules" exist because they are true for a reason. Thinking carefully about what your quantified sentences say will prevent you from negating incorrectly, and you should never apply the rules mindlessly.

Remark 5.4.4. The procedure for checking a sentence is false is now as follows.

- Negate the sentence.
- Verify that the negated sentence is true.

Example 5.4.5. The following sentence is false.

$$\forall x \in \mathbb{R}, \ (x > -10 \implies x \geq 10).$$

Proof. The negation of the sentence is

$$\exists x \in \mathbb{R} : \ (x > -10 \text{ and } x < 10).$$

This is true, because we can verify it as follows.

1. Let $x = 0$.
2. Because $-10 < 0 \leq 10$, we see that $(x > -10 \text{ and } x < 10)$ is true.

$\square$

5.5 Negating a sentence with parentheses carefully

The most frequent reason for negating a sentence is to check that the original sentence is false, but in this section I negate a load of true sentences… because I'm weird. They were just the examples that I thought up.

Example 5.5.1. The negation of the sentence

$$\forall x \in \mathbb{R},\ x > 2 \implies \Big[(x^2 > 4) \text{ and } (x > 0)\Big]$$

is

$$\exists x \in \mathbb{R}:\ x > 2 \text{ and } \Big[(x^2 \le 4) \text{ or } (x \le 0)\Big].$$

Example 5.5.2. The negation of the sentence

$$\forall x \in \mathbb{R},\ x > 0 \implies \Big[\forall y \in \mathbb{R},\ x + y > y\Big]$$

is

$$\exists x \in \mathbb{R}:\ x > 0 \text{ and } \Big[\exists y \in \mathbb{R}:\ x + y \le y\Big].$$

Example 5.5.3. The negation of the sentence

$$\forall x \in \mathbb{R},\ (x > 0) \text{ or } (x > 1 \implies x = 0)$$

is

$$\exists x \in \mathbb{R}:\ (x \le 0) \text{ and } (x > 1 \text{ and } x \ne 0).$$

Example 5.5.4. The negation of the sentence

$$\forall a \in \mathbb{R} \setminus \{0\},\ \forall b \in \mathbb{R},\ \forall c \in \mathbb{R},\ \Big(\exists x \in \mathbb{R}:\ ax^2 + bx + c = 0\Big) \implies b^2 - 4ac \ge 0$$

is given by

$$\exists a \in \mathbb{R} \setminus \{0\}:\ \exists b \in \mathbb{R}:\ \exists c \in \mathbb{R}:\ \Big(\exists x \in \mathbb{R}:\ ax^2 + bx + c = 0\Big) \text{ and } b^2 - 4ac < 0.$$

Recall that the first sentence says that if a real quadratic has a real root, then its discriminant is greater than or equal to 0, a true statement. So the negation is false.

A related example which should be compared with the previous one carefully is as follows.

Example 5.5.5. The negation of the sentence

$$\forall a \in \mathbb{R} \setminus \{0\},\ \forall b \in \mathbb{R},\ \forall c \in \mathbb{R},\ \exists x \in \mathbb{R}:\ ax^2 + bx + c = 0 \implies b^2 - 4ac \ge 0$$

is given by

$$\exists a \in \mathbb{R} \setminus \{0\}:\ \exists b \in \mathbb{R}:\ \exists c \in \mathbb{R}:\ \forall x \in \mathbb{R},\ ax^2 + bx + c = 0 \text{ and } b^2 - 4ac < 0.$$

The first sentence is far more unusual. It is true: a large enough x will make $ax^2 + bx + c$ nonzero, and then the truth table for arrow applies. But the way it is written is not very helpful. A more direct statement related to why it is true is that

$$\forall a \in \mathbb{R} \setminus \{0\},\ \forall b \in \mathbb{R},\ \forall c \in \mathbb{R},\ \exists x \in \mathbb{R}:\ ax^2 + bx + c \ne 0.$$

Example 5.5.6. The negation of the sentence

$$\forall a \in \mathbb{R} \setminus \{0\},\ \forall b \in \mathbb{R},\ \forall c \in \mathbb{R},\ b^2 - 4ac \ge 0 \implies \Big(\exists x \in \mathbb{R}:\ ax^2 + bx + c = 0\Big)$$

is given by

$$\exists a \in \mathbb{R} \setminus \{0\}:\ \exists b \in \mathbb{R}:\ \exists c \in \mathbb{R}:\ b^2 - 4ac \ge 0 \text{ and } \Big(\forall x \in \mathbb{R},\ ax^2 + bx + c \ne 0\Big).$$

5.6 Exercises

1. Negate the following sentences.

   (a) $\exists x \in \mathbb{R}: \ \forall y \in \mathbb{R}, \ \exists z \in \mathbb{R}: \ x^2 + y^2 = z^2.$

   (b) $\forall x \in \mathbb{R}, \ \forall y \in \mathbb{R}, \ \Big(x \leq y \implies \Big(\exists z \in \mathbb{R}: \ x + z^2 = y \Big) \Big).$

   (c)

$$\forall n_1 \in \mathbb{Z}, \ \forall n_2 \in \mathbb{Z}, \ \Big[\Big(\exists k_1 \in \mathbb{Z}: \ n_1 = 2k_1 \Big) \text{ and } \Big(\exists k_2 \in \mathbb{Z}: \ n_2 = 2k_2 \Big) \Big] \implies \Big(\exists k \in \mathbb{Z}: \ n_1 + n_2 = 2k \Big).$$

2. For each of the following sentences, either

   - verify it using the the verification procedure of 5.2,
   - or negate it using the rules of 5.4, and then verify its negation.

   (a) $\forall n \in \mathbb{N}, \ \exists m \in \mathbb{N}: \ n < m.$

   (b) $\exists m \in \mathbb{N}: \ \forall n \in \mathbb{N}, \ n < m.$

   (c) $\forall n \in \mathbb{N}, \ \exists m \in \mathbb{N}: \ n > m.$

   (d) $\forall n \in \mathbb{N}, \ n > 1 \implies (\exists m \in \mathbb{N}: \ n > m).$

   (e) $\exists m \in \mathbb{N}: \ \forall n \in \mathbb{N}, \ (n > 1 \implies n > m).$

   Think carefully about the differences in each of the sentences. Each of your proofs will use a property of $\mathbb{N}$. Try and communicate these properties to a friend in more colloquial language.

3. For each of the sentences $\mathscr{S}$ in exercise 3.8, verify $\mathscr{S}$ or its negation NOT($\mathscr{S}$).

4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x, y) = (y - x^2)(y - 2x^2)$. Consider the following sentences.

$$\forall \theta \in [0, \pi), \ \exists \delta > 0: \ \forall r \in \mathbb{R}, \ (0 < |r| < \delta \implies f(r\cos\theta, r\sin\theta) > f(0, 0)).$$

$$\exists \delta > 0: \ \forall \theta \in [0, \pi), \ \forall r \in \mathbb{R}, \ (0 < |r| < \delta \implies f(r\cos\theta, r\sin\theta) > f(0, 0)).$$

   One is true; one is false.

   (a) Figure out which is which.

   (b) Prove that the true sentence is true.

   (c) Prove that the false sentence is false by negating it, and proving its negation is true.

5.7 Solutions to some exercises

1. (a) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}:\ \forall z \in \mathbb{R},\ x^2 + y^2 \neq z^2$.

   (b) $\exists x \in \mathbb{R}:\ \exists y \in \mathbb{R}:\ x \leq y \ \text{ and } \ \left(\forall z \in \mathbb{R},\ x + z^2 \neq y\right)$.

   (c)

$$\exists n_1 \in \mathbb{Z}:\ \exists n_2 \in \mathbb{Z}:\ \left[\left(\exists k_1 \in \mathbb{Z}:\ n_1 = 2k_1\right) \text{ and } \left(\exists k_2 \in \mathbb{Z}:\ n_2 = 2k_2\right)\right]$$
$$\text{and } \left(\forall k \in \mathbb{Z},\ n_1 + n_2 \neq 2k\right).$$

2. (a) It's true because we can always add one to get a bigger natural number.

   The proof looks like:

   i. Let $n \in \mathbb{N}$.
   ii. Let $m = n + 1$ which is a natural number.
   iii. Then $n < m$.

   (b) It's false because there is not a natural number bigger than all natural numbers, and this is because no natural number is bigger than itself.

   The negation is $\forall m \in \mathbb{N},\ \exists n \in \mathbb{N}:\ n \geq m$.

   The proof of this is:

   i. Let $m \in \mathbb{N}$.
   ii. Let $m = n$ which is a natural number.
   iii. We have $n = m$. In particular, we have $n \geq m$.

   (c) It is false because 1 is the smallest natural number, so has no naturals strictly smaller than it.

   The negation is $\exists n \in \mathbb{N}:\ \forall m \in \mathbb{N},\ n \leq m$.

   The proof of this is:

   i. Let $n = 1$. We have $n \in \mathbb{N}$.
   ii. Let $m \in \mathbb{N}$.
   iii. We have $n = 1 \leq m$.

   (d) It is true because 1 is a natural number; thus, natural numbers bigger than 1 always have this natural number below them.

   The proof looks like:

   i. Let $n \in \mathbb{N}$.
   ii. Suppose $n > 1$. We must verify $(\exists m \in \mathbb{N}:\ n > m)$.
      A. Let $m = 1$.
      B. Then $n > 1 = m$.

(e) It is true because 1 is a natural number; thus, natural numbers bigger than 1 always have this natural number below them.

The proof looks like:

i. Let $m = 1$. We have $m \in \mathbb{N}$.

ii. Let $n \in \mathbb{N}$.

iii. Suppose $n > 1$. We must verify $n > m$. This is because $n > 1 = m$.

3. Omitted.

4. (a) The first is true. The second is false.

You can deduce this without thinking about the sentences very much at all. If the second sentence was true, the first would have to be true as well true; this is because if you can find a δ *not* depending on θ, you can find a δ depending (trivially) on θ. However, I told you only one sentence is true, so the second must be false.

(b)
- Let $\theta \in [0, \pi)$.
- There are three cases.
  - $\theta = 0$. Let $\delta = 1$.
  - $\theta = \frac{\pi}{2}$. Let $\delta = 1$.
  - $\theta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Let $\delta = \frac{\sin\theta}{2\cos^2\theta}$.
    The condition $\theta \in (0, \pi)$ implies that $\sin\theta > 0$.
    The condition $\theta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ implies $\cos\theta \neq 0$. Thus, $\cos^2\theta > 0$.
    All of this shows δ is well-defined and strictly positive.
- Let $r \in \mathbb{R}$.
- We must show that $(0 < |r| < \delta \implies f(r\cos\theta, r\sin\theta) > f(0,0))$ is true.
  In the cases that $r = 0$ or $|r| \geq \delta$, it is trivially true.
  Suppose $0 < |r| < \delta$. Since $f(0,0) = 0$, we must show that $f(r\cos\theta, r\sin\theta) > 0$.
  - When $\theta = 0$ we have $f(r\cos\theta, r\sin\theta) = f(r, 0) = 2r^4 > 0$.
  - When $\theta = \frac{\pi}{2}$ we have $f(r\cos\theta, r\sin\theta) = f(0, r) = r^2 > 0$.
  - When $\theta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$, we have

$$f(r\cos\theta, r\sin\theta) = r^2(\sin\theta - r\cos^2\theta)(\sin\theta - 2r\cos^2\theta).$$

    * Case 1: $r > 0$.
      Then $0 < r < \frac{\sin\theta}{2\cos^2\theta}$ and so $r\cos^2\theta \leq 2r\cos^2\theta < \sin\theta$. Thus,

$$\sin\theta - r\cos^2\theta > 0 \quad \text{and} \quad \sin\theta - 2r\cos^2\theta > 0.$$

      Since $r \neq 0$, this shows that $f(r\cos\theta, r\sin\theta) > 0$.
    * Case 2: $r < 0$.
      The following inequalities hold (more trivially than above).

$$\sin\theta - r\cos^2\theta > 0 \quad \text{and} \quad \sin\theta - 2r\cos^2\theta > 0$$

      Since $r \neq 0$, this shows that $f(r\cos\theta, r\sin\theta) > 0$.

(c) We must show that the following sentence is true.

$$\forall \delta > 0,\ \exists \theta \in [0, \pi) :\ \exists r \in \mathbb{R} :\ 0 < |r| < \delta \text{ and } f(r\cos\theta, r\sin\theta) \leq 0.$$

- Let $\delta > 0$.
- To specify θ and r we first let $x = \min\{1, \frac{\delta}{2}\}$. Note that $x > 0$.
- Let $\theta = \arctan(x)$. Since $x > 0$, we have $\theta \in (0, \frac{\pi}{2})$. In particular, $\theta \in [0, \pi)$.
- Let $r = x\sqrt{1 + x^2}$. Then $r \in \mathbb{R}$.
- One sees that

$$0 < x \leq x\sqrt{1 + x^2} \leq x\sqrt{1 + 1} = x\sqrt{2} \leq \frac{\delta}{2} \cdot \sqrt{2} < \delta$$

  so that $0 < |r| < \delta$.
- Also, $f(r\cos\theta, r\sin\theta) = f(x, x^2) = 0$. In particular, $f(r\cos\theta, r\sin\theta) \leq 0$.

6 Connection to part 1 •

6.1 Questions

1. Fill in the "BLANKS" in the following definition with quantified sentences.

   I expect each "BLANK" to have two quantifiers.

   Definition. Suppose X and Y are sets and that $f : X \longrightarrow Y$ is a function.

   (a) We say f is *injective* **iff** "BLANK."
   (b) We say f is *surjective* **iff** "BLANK."
   (c) We say f is *bijective* **iff** f is injective and surjective.

2. Fill in the "BLANKS" in the following definition with quantified sentences.

   I expect the n-th definition to use n quantifiers.

   Definition. A relation $\mathcal{R}$ on a set X is said to be:

   - *reflexive* **iff** "BLANK;"
   - *symmetric* **iff** "BLANK;"
   - *transitive* **iff** "BLANK."

3. Look back on example 19.9 of part 1.

   Definition. Fix $n \in \mathbb{N}$. We can define a relation on $\mathbb{Z}$ by declaring $a \sim_n b$ **iff**

   "BLANK."

4. Let X be a set.

   (a) Write down the quantified sentence that says $f : X \longrightarrow \mathcal{P}(X)$ is surjective.
   (b) Write down its negation.
   (c) Verify the negation.

6.2 Answers

1. (a) We say f is *injective* **iff**

$$\forall x_1 \in X,\ \forall x_2 \in X,\ f(x_1) = f(x_2) \implies x_1 = x_2.$$

   (b) We say f is *surjective* **iff**

$$\forall y \in Y,\ \exists x \in X:\ f(x) = y.$$

2. A relation $\mathcal{R}$ on a set X is said to be:

   - *reflexive* **iff** $\forall x \in X,\ x\mathcal{R}x$.
   - *symmetric* **iff** $\forall x_1 \in X,\ \forall x_2 \in X,\ x_1\mathcal{R}x_2 \implies x_2\mathcal{R}x_1$.
   - *transitive* **iff** $\forall x_1 \in X,\ \forall x_2 \in X,\ \forall x_3 \in X,\ (x_1\mathcal{R}x_2 \text{ and } x_2\mathcal{R}x_3) \implies x_1\mathcal{R}x_3$.

3. Fix $n \in \mathbb{N}$. For $a, b \in \mathbb{Z}$, $a \sim_n b$ **iff**

$$\exists d \in \mathbb{Z}:\ a - b = dn.$$

4. (a) $\forall A \in \mathcal{P}(X),\ \exists x \in X:\ f(x) = A$.

   (b) $\exists A \in \mathcal{P}(X):\ \forall x \in X,\ f(x) \neq A$.

   (c) i. Let $A = \{x \in X:\ x \notin f(x)\}$.

   ii. Let $x \in X$.

   iii. We must show that $f(x) \neq A$. This is done as in part 1 of the notes.

7 Pre-analysis exercises •

7.1 Questions

1. (a) Negate the following sentence.

$$\forall n \in \mathbb{N}, \quad \frac{9999}{n^2 - 750} \leq \frac{10000}{n^2}.$$

(b) Using the negation you found in part (a), prove that the sentence of part (a) is false.

(c) Prove that the following sentence is true.

$$\exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \left(n \geq N \implies \frac{9999}{n^2 - 750} \leq \frac{10000}{n^2} \right).$$

(d) What is the point in replacing $\forall n \in \mathbb{N}$ with a clause like the following one?

$$\exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \left(n \geq N \implies \dots \right)$$

2. (a) Negate the following sentence.

$$\forall x \in \mathbb{R}, \ 2|x| > 1.$$

(b) Using the negation you found in part (a), prove that the sentence of part (a) is false.

(c) Prove that the following sentence is true.

$$\exists \delta > 0 : \ \forall x \in \mathbb{R}, \left(|x - 1| < \delta \implies 2|x| > 1 \right).$$

("$\exists d > 0$" is shorthand for "$\exists \delta \in \{x \in \mathbb{R} : \ x > 0\}$.")

(d) What is the point in replacing $\forall x \in \mathbb{R}$ with a clause like the following one?

$$\exists \delta > 0 : \ \forall x \in \mathbb{R}, \left(|x - 1| < \delta \implies \dots \right)$$

7.2 Answers

1. (a) $\exists n \in \mathbb{N}: \ \frac{9999}{n^2-750} > \frac{10000}{n^2}$.

   (b) Let $n = 100$. Then $\frac{9999}{n^2-750} = \frac{9999}{10000-750} = \frac{9999}{9250} > 1 = \frac{10000}{n^2}$.

   (c) Let $N = 3000$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. We wish to show $\frac{9999n^2}{n^2-750} \leq \frac{10000}{n^2}$. We have $n \geq 3 \cdot 10^3$. So $n^2 \geq 9 \cdot 10^6 \geq 75 \cdot 10^5$. Adding $9999n^2$ to both sides gives

   $$10^4 \cdot n^2 \geq 9999n^2 + 75 \cdot 10^5.$$

   Thus, $0 < 9999n^2 \leq 10^4 \cdot (n^2 - 750)$, which gives $\frac{9999n^2}{n^2-750} \leq \frac{10000}{n^2}$.

   (d) If $P(n)$ is a propositional function depending on a natural number n, then

   $$\exists N \in \mathbb{N}: \ \forall n \in \mathbb{N}, \ \Big(n \geq N \implies P(n)\Big)$$

   means that $P(n)$ holds "eventually." A more precise way of saying "eventually" would be "for large enough n." A more precise way of saying this is just to write out or say the quantified sentence.

2. (a) $\exists x \in \mathbb{R}: \ 2|x| \leq 1$.

   (b) Let $x = \frac{1}{2} \ldots$

   (c) i. Let $\delta = \frac{1}{2}$.

   ii. Let $x \in \mathbb{R}$.

   iii. Suppose that $|x-1| < \delta$. We want to show that $2|x| > 1$. Using the triangle ineqality, we obtain

   $$1 = |(1-x)+x| \leq |1-x| + |x| = |x-1| + |x| < \delta + |x| = \frac{1}{2} + |x|.$$

   Subtracting a half from both sides gives $|x| > \frac{1}{2}$, i.e. $2|x| > 1$.

   (d) If $P(x)$ is a propositional function depending on a real number number x, then

   $$\exists \delta > 0: \ \forall x \in \mathbb{R}, \ \Big(|x-1| < \delta \implies P(x)\Big)$$

   means that $P(x)$ holds "near 1." A more precise way of saying "near 1" would be "for x close enough to 1." A more precise way of saying this is just to write out or say the quantified sentence.

8 Sequences and the definition of convergence •

"Black (1)
then (1)
white are (2)
all I see (3)
in my in-fan-cy, (5)
red and yel-low then came to be (8)..."

Tool, Los Angeles, 2001.

8.1 Sequences

A sequence is what you think it is.

Here is an informal definition of a sequence. The formal definition is on the next page.

Definition 8.1.1 (Informal)**.** A *sequence* is an infinite list of numbers.

Here are some examples:

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots,\ n,\ \ldots$$

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \ldots,\ \frac{1}{n},\ \ldots$$

$$-1,\ 1,\ -1,\ 1,\ -1,\ \ldots,\ (-1)^n,\ \ldots$$

$$2,\ \left(\frac{3}{2}\right)^2,\ \left(\frac{4}{3}\right)^3,\ \left(\frac{5}{4}\right)^4,\ \left(\frac{6}{5}\right)^5, \ldots, \left(1+\frac{1}{n}\right)^n, \ldots$$

$$1,\ 1,\ 2,\ 3,\ 5,\ \ldots,\ F_n = F_{n-1} + F_{n-2},\ \ldots$$

$$\frac{1}{1},\ \frac{2}{1},\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \ldots,\ \frac{F_{n+1}}{F_n},\ \ldots$$

We frequently write s_n for the n-th term in a sequence, and so we could express the first four sequences above, concisely, by saying

$$s_n = n,\ s_n = \frac{1}{n},\ s_n = (-1)^n,\ s_n = \left(1+\frac{1}{n}\right)^n.$$

(Is there a formula for the n-th Fibonacci number which is not iterative? Can you find evidence of the Fibonacci sequence in the Tool song "Lateralus," other than that highlighted in the quote at the start of this section?)

Notation 8.1.2. To indicate an abstract sequence we use parentheses: (s_n). Sometimes we may want to emphasize that the list is indexed by natural numbers and we write $(s_n)_{n\in\mathbb{N}}$. We might also write $(s_n)_{n=1}^{\infty}$.

Definition 8.1.3 (Formal)**.** A *sequence* is a function $s : \mathbb{N} \longrightarrow \mathbb{R}$. When $n \in \mathbb{N}$, we write s_n for $s(n)$. Instead of writing $s : \mathbb{N} \longrightarrow \mathbb{R}$, we often write (s_n), $(s_n)_{n\in\mathbb{N}}$, or $(s_n)_{n=1}^{\infty}$.

Notation 8.1.4. It is convenient to be able to start a sequence with s_m, where m might be greater than 1. In this case we write $(s_n)_{n=m}^{\infty}$. So, for example,

$$\left(\frac{1}{n}\right)_{n=8}^{\infty} = \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \dots.$$

8.2 Convergence of sequences

Here's the first major definition concerning sequences.

Definition 8.2.1. Suppose $(s_n)_{n=1}^{\infty}$ is a sequence.

We say that (s_n) *converges* **iff** the following sentence is true.

$$\exists L \in \mathbb{R} : \ \forall \epsilon > 0, \ \exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N \implies |s_n - L| < \epsilon.$$

Remark 8.2.2. The quanitifier "$\forall \epsilon > 0$" is shorthand for "$\forall \epsilon \in \{x \in \mathbb{R} : \ x > 0\}$."

First, let me show you a sequence which converges and a sequence which does not converge, and proofs of those facts without any explanation of where the proofs come from.

Theorem 8.2.3. *The sequence* $(\frac{1}{n})_{n=1}^{\infty}$ *converges.*

Proof. By definition, we must verify that the following sentence is true.

$$\exists L \in \mathbb{R} : \ \forall \epsilon > 0, \ \exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N \implies \left|\frac{1}{n} - L\right| < \epsilon.$$

1. Let $L = 0$.
2. Let $\epsilon > 0$.
3. Let $N = \lceil \frac{1}{\epsilon} \rceil + 1$.

   Here $\lceil - \rceil$ denotes the ceiling function. It rounds up to the nearest integer. Note that it never rounds down. For example,
   $$\left\lceil \frac{9}{4} \right\rceil = 3 = \lceil 3 \rceil.$$
   This is useful frequently, so remember it.
4. Let $n \in \mathbb{N}$.
5. We must verify $(n \geq N \implies |\frac{1}{n} - L| < \epsilon)$.

   (a) Case 1: $n < N$.
   $n \geq N$ is false, so the truth table for "$\implies$" says $(n \geq N \implies |\frac{1}{n} - L| < \epsilon)$ is true.

(b) Case 2: $n \geq N$.
We must show that $|\frac{1}{n} - L| < \epsilon$.
We have $N = \lceil \frac{1}{\epsilon} \rceil + 1 > \frac{1}{\epsilon}$ and also

$$\left|\frac{1}{n} - L\right| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

□

8.3 Divergence of sequences

Definition 8.3.1. Suppose $(s_n)_{n=1}^{\infty}$ is a sequence.
We say that (s_n) *diverges* **iff** (s_n) does not converge, i.e. **iff** the following sentence is true.

$$\forall L \in \mathbb{R},\ \exists \epsilon > 0 :\ \forall N \in \mathbb{N},\ \exists n \in \mathbb{N} :\ n \geq N \text{ and } |s_n - L| \geq \epsilon.$$

Theorem 8.3.2. *The sequence* $((-1)^n)_{n=1}^{\infty}$ *diverges.*

Proof. By definition, we must verify that the following sentence is true.

$$\forall L \in \mathbb{R},\ \exists \epsilon > 0 :\ \forall N \in \mathbb{N},\ \exists n \in \mathbb{N} :\ n \geq N \text{ and } |(-1)^n - L| \geq \epsilon.$$

1. Let $L \in \mathbb{R}$.

2. Let $\epsilon = 1$.

3. Let $N \in \mathbb{N}$.

4. (a) Case 1: $L \geq 0$. Let $n = 2N + 1$.
   (b) Case 2: $L < 0$. Let $n = 2N$.

5. We must verify that $(n \geq N \text{ and } |(-1)^n - L| \geq \epsilon)$ is true.

   (a) Case 1: $L \geq 0$.
   $n = 2N + 1 = N + (N + 1) \geq N$. Also,

   $$|(-1)^n - L| = |(-1)^{2N+1} - L| = |-1 - L| = 1 + L \geq 1 = \epsilon.$$

   (b) Case 2: $L < 0$.
   $n = 2N = N + N \geq N$. Also,

   $$|(-1)^n - L| = |(-1)^{2N} - L| = |1 - L| = 1 - L \geq 1 = \epsilon.$$

□

8.4 How to spend your time

In terms of your learning, your current goals should be as follows.

- Make sure you are happy with all of the quantifier verifications of the previous sections.
- Make sure you are happy with the correctness of the last two proofs given, even if you have no clue where the idea for them came from.
-
  - Look at the previous two proofs.
  - Figure out which parts of the proof are mindless, and will be the same for any sequence.
  - Figure out which bits require thought. Think about these parts and see if you can figure out what thinking I did to come up with them.

  You should spend a long time on this step before going on to the next one, either until you think that you have figured it out, or until two hours has passed.
- Try and understand what the definition of convergence is saying. It is a wonderful and clever definition. It encodes everything you could ever want.

  You should spend a long time on this step before going on to the next one, either until you think that you have figured it out, or until two hours has passed.
- Read further and understand my description of where the proofs came from, and what the definition of convergence is saying.

8.5 Constructing the proof of theorem 8.2.3

Let's revisit theorem 8.2.3, the fact that $(\frac{1}{n})_{n=1}^{\infty}$ converges.

We must verify that the following sentence is true.

$$\exists L \in \mathbb{R} : \ \forall \epsilon > 0, \ \exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N \implies \left| \frac{1}{n} - L \right| < \epsilon.$$

A lot of the proof is fairly mindless and it is likely that writing what follows is a step in the right direction.

1. Let $L =$ BLAH.
2. Let $\epsilon > 0$.
3. Let $N =$ BLAH.
4. Let $n \in \mathbb{N}$.
5. We must verify $(n \geq N \implies |\frac{1}{n} - L| < \epsilon)$.
   (a) Case 1: $n < N$.
   $n \geq N$ is false, so the truth table for "$\implies$" says $(n \geq N \implies |\frac{1}{n} - L| < \epsilon)$ is true.
   (b) Case 2: $n \geq N$.
   We must show that $|\frac{1}{n} - L| < \epsilon$. We have
   $$\left| \frac{1}{n} - L \right| = \ldots.$$
   BLAH BLAH BLAH.

We have to fill in the BLAHs. First, we must figure out what L is. What's the significance of L? I hope that after thinking about the definition of convergence and the proofs of theorem 8.2.3 long enough you will have come to the conclusion that L is supposed to be the *limit* of the sequence.

Even if you have not parsed all the quantifiers just yet, you know that the condition $|s_n - L| < \epsilon$ says "s_n is within ϵ of L." The limit of a sequence is precisely the real number that the terms in a sequence get close to, so L has got to be the limit, right?! Let's take a minute to make a definition.

Definition 8.5.1. Suppose $(s_n)_{n=1}^{\infty}$ is a sequence and $L \in \mathbb{R}$.

We say that (s_n) *converges to* L **iff** the following sentence is true.

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N \implies |s_n - L| < \epsilon.$$

In this case L is said to be the *limit* of the sequence (s_n) and we will often express this by writing $\lim_{n \to \infty} s_n = L$ or, if it is not confusing, $\lim s_n = L$.

Remark 8.5.2.

1. If you look at the quantified sentence in the previous definition, you will see that L has not been quantified over. This is okay because L was declared in the preamble. By the time we state the quantified sentence, L is a *constant* real number.

2. (a) If (s_n) converges, then there is some $L \in \mathbb{R}$ such that (s_n) converges to L.

   (b) If (s_n) converges to a number $L \in \mathbb{R}$, then (s_n) converges.

   These statements are trivial from the definitions, but in this early stage of analysis, you should check that you understand them 100%.

I hope that while taking math 31B or a similar class you were told that $\lim_{n\to\infty} \frac{1}{n} = 0$. We are now *proving* this fact. This is the reason that I took L to be 0 in my proof.

Now we can improve upon the beginning of the proof written above a little bit.

1. Let $L = 0$.

5. (b) Case 2: $n \geq N$.
   We must show that $|\frac{1}{n} - L| < \epsilon$. We have

$$\left|\frac{1}{n} - L\right| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} \dots$$

Now to N.

- **The purpose of N is to specify how far along in the sequence we have to go before terms in the sequence are within ϵ of the limit L.**
- N will almost always depend on ϵ.
- If ϵ is small, N will almost always be large. Formulae like $N = \epsilon$ are stupid.

In the case that we're considering we see that beyond N, $\frac{1}{n}$ is within $\frac{1}{N}$ of 0, and so we just need $\frac{1}{N} < \epsilon$, i.e. $N > \frac{1}{\epsilon}$. I've decided to go back to my Oxford routes and demand that N be a natural number. The textbook says otherwise. Sorry about this. It'll be an exercise later to show that this choice doesn't matter. $\lceil \frac{1}{\epsilon} \rceil + 1$ is the simplest way to write down an explicit natural number bigger than $\frac{1}{\epsilon}$.

We've now completed the proof.

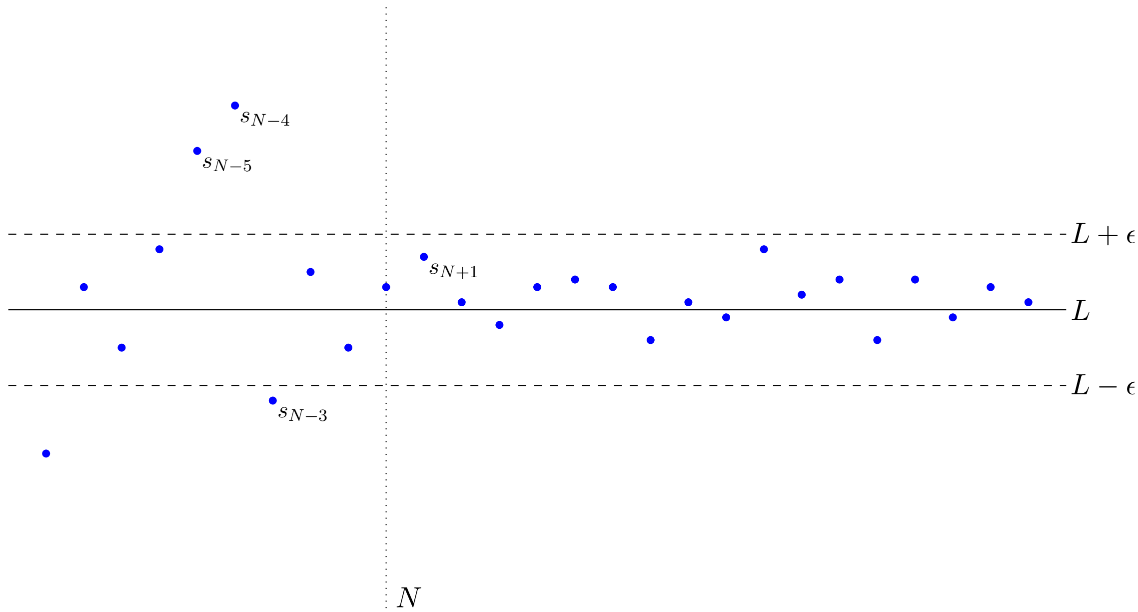
3. Let $N = \lceil \frac{1}{\epsilon} \rceil + 1$.

5. (a) Case 2: $n \geq N$.
   We must show that $|\frac{1}{n} - L| < \epsilon$.
   We have $N = \lceil \frac{1}{\epsilon} \rceil + 1 > \frac{1}{\epsilon}$ and also

$$\left|\frac{1}{n} - L\right| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

8.6 Motivating the definition of convergence

A picture is worth a thousand words.



A sequence of real numbers (s_n) converges to a real number L, the limit, if

- whenever a small positive quantity ϵ is specified
- by going far enough along in the sequence
- we ensure that the terms of the sequence differ from the limit by less than the small quantity specified.

The first bullet point is said mathematically as $\forall \epsilon > 0$.
The last two are said mathematically as follows.

- $\exists N \in \mathbb{N}$ (there exists a number telling us how far we need to go)
- $: \ \forall n \in \mathbb{N}, \ n \geq N \implies$ (if we go this far along in the sequence)
- $|s_n - L| < \epsilon$ (then the terms of the sequence are within ϵ of L).

8.7 Another sequence that converges

Theorem 8.7.1. *The sequence* $(\frac{n}{n^2-101})_{n=1}^{\infty}$ *converges.*

Before writing the proof let's think what it must look like. I hope that you can see that the limit of this sequence will be 0. Thus, our proof is going to look as follows.

We must verify that the following sentence is true.

$$\exists L \in \mathbb{R}: \ \forall \epsilon > 0, \ \exists N \in \mathbb{N}: \ \forall n \in \mathbb{N}, \ n \geq N \implies \left|\frac{n}{n^2-101} - L\right| < \epsilon.$$

1. Let $L = 0$.
2. Let $\epsilon > 0$.
3. Let $N =$ BLAH.
4. Let $n \in \mathbb{N}$.
5. We must verify $(n \geq N \implies |\frac{n}{n^2-101} - L| < \epsilon)$.
   (a) Case 1: $n < N$.
   $n \geq N$ is false, so the truth table for " $\implies$ " says $(n \geq N \implies |\frac{n}{n^2-101} - L| < \epsilon)$ is true.
   (b) Case 2: $n \geq N$.
   We must show that $|\frac{n}{n^2-101} - L| < \epsilon$. We have

   $$\left|\frac{n}{n^2-101} - L\right| = \left|\frac{n}{n^2-101}\right| = \dots.$$

   BLAH BLAH BLAH.

Now I get to say more about N. The clause

$$\exists N \in \mathbb{N}: \ \forall n \in \mathbb{N}, \ n \geq N \implies \left|\frac{n}{n^2-101}\right| < \epsilon$$

says that $|\frac{n}{n^2-101}| < \epsilon$ is true as long as n is sufficiently large, and N is a natural number quantifying exactly how large n needs to be.

When considering the expression $|\frac{n}{n^2-101}|$ my first thought is "is $\frac{n}{n^2-101}$ positive?" The answer is, "not necessarily, but as long as n is sufficiently large." We can improve on this and say that "it is positive as long as $n > \sqrt{101}$."

$$\forall n \in \mathbb{N}, \ n > \sqrt{101} \implies \frac{n}{n^2-101} > 0.$$

We can now pretty much work under the assumption that $\frac{n}{n^2-101}$ positive. But by saying "pretty much," I mean that we have to remember the conditions under which it actually holds: $n > \sqrt{101}$.

My next thought is that $\frac{n}{n^2-101}$ should behave somewhat like $\frac{1}{n}$. A useful inequality towards a proof would be

$$\forall n \in \mathbb{N},\ \frac{n}{n^2-101} \le \frac{1}{n}$$

but this is UTTER NONSENSE!! In fact, the following sentence is true

$$\forall n \in \mathbb{N},\ n > \sqrt{101} \implies \frac{n}{n^2-101} > \frac{1}{n}.$$

The inequality goes the wrong way. Instead, we note that

$$\forall n \in \mathbb{N},\ n \ge \sqrt{202} \implies 0 \le \frac{n}{n^2-101} \le \frac{2}{n} \tag{8.7.2}$$

This is because (for $n \in \mathbb{N}$) the condition $0 \le \frac{n}{n^2-101} \le \frac{2}{n}$ is equivalent to $202 \le n^2$ by basic algebra. The idea I had here was "$\frac{1}{n}$ doesn't give a correct inequality, but maybe $\frac{2}{n}$ does." In the end, the 2 made all the difference.

We finally note that the following sentence is true.

$$\forall n \in \mathbb{N},\ n > \frac{2}{\epsilon} \implies \frac{2}{n} < \epsilon. \tag{8.7.3}$$

Equations (8.7.2) and (8.7.3) suggest taking $N = \lceil \max\{\sqrt{202}, \frac{2}{\epsilon}\}\rceil + 1$.

Proof of theorem 8.7.1. We must verify that the following sentence is true.

$$\exists L \in \mathbb{R} :\ \forall \epsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \ge N \implies \left| \frac{n}{n^2-101} - L \right| < \epsilon.$$

1. Let $L = 0$.
2. Let $\epsilon > 0$.
3. Let $N = \lceil \max\{\sqrt{202}, \frac{2}{\epsilon}\}\rceil + 1$.
4. Let $n \in \mathbb{N}$.
5. We must verify $(n \ge N \implies |\frac{n}{n^2-101} - L| < \epsilon)$.
   (a) Case 1: $n < N$.
   $n \ge N$ is false, so the truth table for "$\implies$" says $(n \ge N \implies |\frac{n}{n^2-101} - L| < \epsilon)$ is true.
   (b) Case 2: $n \ge N$.
   Since $N = \lceil \max\{\sqrt{202}, \frac{2}{\epsilon}\}\rceil + 1 > \sqrt{101}$, we have $n > \sqrt{101}$ and $n^2 - 101 > 0$.
   Since $N = \lceil \max\{\sqrt{202}, \frac{2}{\epsilon}\}\rceil + 1 > \sqrt{202}$, we have $n \ge \sqrt{202}$ and $\frac{n}{n^2-101} \le \frac{2}{n}$.
   Since $N = \lceil \max\{\sqrt{202}, \frac{2}{\epsilon}\}\rceil + 1 > \frac{2}{\epsilon}$, we have $n > \frac{2}{\epsilon}$.
   Thus,
   $$\left| \frac{n}{n^2-101} - L \right| = \left| \frac{n}{n^2-101} \right| = \frac{n}{n^2-101} \le \frac{2}{n} < \epsilon.$$

$\square$

8.8 Constructing the proof of theorem 8.3.2

Let's revisit theorem 8.3.2, the fact that $((-1)^n)_{n=1}^{\infty}$ diverges.

We must verify that the following sentence is true.

$$\forall L \in \mathbb{R},\ \exists \epsilon > 0 :\ \forall N \in \mathbb{N},\ \exists n \in \mathbb{N} :\ n \geq N \text{ and } |(-1)^n - L| \geq \epsilon.$$

A lot of the proof is fairly mindless and it is likely that writing what follows is a step in the right direction.

1. Let $L \in \mathbb{R}$.
2. Let $\epsilon =$ BLAH.
3. Let $N \in \mathbb{N}$.
4. Let $n =$ BLAH.
5. We must verify that $(n \geq N \text{ and } |(-1)^n - L| \geq \epsilon)$ is true.

   BLAH.

This looks tricky because ϵ is allowed to depend on L, and n is allowed to depend on L, ϵ, and N. That's a lot to juggle.

It turns out that whenever (s_n) is a sequence,

$$\begin{array}{rl} & \text{“}\forall L \in \mathbb{R},\ \exists \epsilon > 0 :\ \forall N \in \mathbb{N},\ \exists n \in \mathbb{N} :\ n \geq N \text{ and } |s_n - L| \geq \epsilon\text{” is true} \\ \text{if and only if} & \text{“}\exists \epsilon > 0 :\ \forall L \in \mathbb{R},\ \forall N \in \mathbb{N},\ \exists n \in \mathbb{N} :\ n \geq N \text{ and } |s_n - L| \geq \epsilon\text{” is true.} \end{array}$$

This is quite a subtle fact. For now, this simplifies things a little. It tells you that ϵ does not need to depend on L; that is the point in swapping the first two quantifiers.

In the case of $((-1)^n)_{n=1}^{\infty}$, my intuition for choosing $\epsilon = 1$ is the following.

- The clause
  $$\forall N \in \mathbb{N},\ \exists n \in \mathbb{N} :\ n \geq N \text{ and } |(-1)^n - L| \geq \epsilon$$
  says that however far along in the sequence we go, we can always find a term in the sequence which is ϵ or further away from L.
- We don't know what L is but we can think about how difficult it would be to prove that the sentence above is true for different Ls.
- Suppose L is 1. Then the even terms of the sequence are 0 away from L. However, the odd terms are 2 away from L. This suggests taking $\epsilon = 2$.
- Suppose L is -1. Then the odd terms of the sequence are 0 away from L. However, the even terms are 2 away from L. This suggests taking $\epsilon = 2$.
- Suppose L is 500. Then the odd terms of the sequence are 501 away from L and the even terms are 499 away from L. This continues to suggest that $\epsilon = 2$ would be a fine choice.

- Suppose L is 0. Then all of the terms are 1 away from L. This suggests that $\epsilon = 1$ would be a better choice.
- For the first three choices of L that we spoke about, we considered the even and odd terms separately. We did not have to do this for the choice $L = 0$. All of this hinted to me that considering the cases $L \geq 0$ and $L < 0$ separately might be useful; there is some symmetry going on.
- Suppose $L \geq 0$. Then the odd terms of the sequence are at least 1 away from L and this continues to suggest that $\epsilon = 1$ would be a fine choice.
- Suppose $L < 0$. Then the even terms of the sequence are at least 1 away from L and this continues to suggest that $\epsilon = 1$ would be a fine choice.

All of the above reasoning suggests taking $\epsilon = 1$. It also suggests that when specifying n we should break up into two cases, when $L \geq 0$ and when $L < 0$.

Our incomplete proof now becomes.

1. Let $L \in \mathbb{R}$.
2. Let $\epsilon = 1$.
3. Let $N \in \mathbb{N}$.
4. (a) Case 1: $L \geq 0$. Let $n =$ BLAH.
   (b) Case 2: $L < 0$. Let $n =$ BLAH.
5. We must verify that ($n \geq N$ and $|(-1)^n - L| \geq \epsilon$) is true.
   (a) Case 1: $L \geq 0$.
   BLAH.
   (b) Case 2: $L < 0$.
   BLAH.

Let's consider how to choose n in the case that $L \geq 0$. When $L \geq 0$, the odd terms of the sequence are at least 1 away from L. This tells us we should choose n to be an odd number. We also want n to be bigger than or equal to N. $2N + 1$ is an odd number bigger than or equal to N.

When $L < 0$, the even terms of the sequence are at least 1 away from L. This tells us we should choose n to be an even number. We also want n to be bigger than or equal to N. $2N$ is an even number bigger than or equal to N.

This allows us to fill in.

4. (a) Case 1: $L \geq 0$. Let $n = 2N + 1$.
   (b) Case 2: $L < 0$. Let $n = 2N$.

We are just left to complete step 5. By this point this is just a matter of writing down everything correctly. However, this step is important and I'll be annoyed when I see nonsense inequalities, or incorrect absolute values, so make sure you get things correct!

8.9 Exercises

1. For each of the following sequences $(s_n)_{n=1}^{\infty}$ prove that (s_n) converges. You should use the definition of convergence given in these notes. No "algebra of limits" or "limit laws" are allowed.

   (a) $s_n = \frac{n}{n+1}$;

   (b) $s_n = \sqrt{\frac{n}{n+1}}$ (you might need the fact that $1 = \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$);

   (c) $s_n = \frac{7n+3}{2n^2-3n-19}$.

   This last part is designed to give you practice at being careful with inequalities; it's much more difficult.
   Note that $\forall n \in \mathbb{N},\ \left|\frac{1}{2n^2-3n-19}\right| \leq \frac{1}{2n^2}$ is false but the following sentence is true.

   $$\exists N_0 \in \mathbb{N}:\ \forall n \in \mathbb{N},\ n \geq N_0 \implies \left|\frac{1}{2n^2-3n-19}\right| \leq \frac{1}{n^2}.$$

   The following sentences are also true. One or two of them might be useful for you.

   $$\begin{aligned} \exists N_1 \in \mathbb{N}:\ \forall n \in \mathbb{N},\ n \geq N_1 &\implies \frac{7}{n} < \frac{\epsilon}{2}; \\ \exists N_2 \in \mathbb{N}:\ \forall n \in \mathbb{N},\ n \geq N_2 &\implies \frac{3}{n^2} < \frac{\epsilon}{2}; \\ \exists N_3 \in \mathbb{N}:\ \forall n \in \mathbb{N},\ n \geq N_3 &\implies \frac{10}{n} < \epsilon. \end{aligned}$$

   It'd be better if your proof made no reference to N_0, N_1, N_2, or N_3. Figure out why the sentences above are true, and use the numbers you obtain from your thinking.
   You should justify in your proof why your chosen numbers work.

   Solution: The hardest bit (not the whole proof) to write down carefully in (c)….
   We have $n \geq 7$. So $n - \frac{3}{2} \geq 5$. Thus, $(n-\frac{3}{2})^2 \geq 25 \geq \frac{85}{4}$. Expanding gives

   $$n^2 - 3n + \frac{9}{4} \geq \frac{85}{4}.$$

   This gives $n^2 - 3n - 19 \geq 0$ and adding n^2 to both sides gives $2n^2 - 3n - 19 \geq n^2$.
   Since $n^2 > 0$, this shows $2n^2 - 3n - 19 > 0$, so $\frac{1}{|2n^2-3n-19|} = \frac{1}{2n^2-3n-19}$.
   Moreover, $2n^2 - 3n - 19 \geq n^2 > 0$ allows us to see that $0 < \frac{1}{2n^2-3n-19} \leq \frac{1}{n^2}$.
   In conclusion, we now know that $\frac{1}{|2n^2-3n-19|} \leq \frac{1}{n^2}$.

2. Prove that the following sequences diverge by using the definition given in the previous section.

   (a) $(n)_{n=1}^{\infty}$;

   (b) $((-1)^n n)_{n=1}^{\infty}$.

8.10 Concise solutions to previous exercises

1. (a) Let $L = 1$.
   Let $\epsilon > 0$.
   Let $N = \lceil \frac{1}{\epsilon} \rceil$.
   Let $n \in \mathbb{N}$.
   Suppose $n \geq N$.
   Then $\left|\frac{n}{n+1} - L\right| = \left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} \leq \epsilon$.

   (b) Let $L = 1$.
   Let $\epsilon > 0$.
   Let $N = \lceil \frac{1}{\epsilon} \rceil$.
   Let $n \in \mathbb{N}$.
   Suppose $n \geq N$.
   We have

$$\left|\sqrt{\frac{n}{n+1}} - L\right| = \left|\sqrt{\frac{n}{n+1}} - 1\right| = \left|\frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n+1}}\right| = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1}}.$$

   Moreover, "multiplying by the conjugate" allows us to see

$$\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{n+1}(\sqrt{n+1} + \sqrt{n})} < \frac{1}{\sqrt{n+1}\sqrt{n+1}} = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} \leq \epsilon.$$

   (c) Let $L = 0$.
   Let $\epsilon > 0$.
   Let $N = \max\{7, \lceil \frac{10}{\epsilon} \rceil\}$.
   Let $n \in \mathbb{N}$.
   Suppose $n \geq N$.
   We wish to show that $\left|\frac{7n+3}{2n^2-3n-19}\right| < \epsilon$.
   First, we go on a detour to show that $\frac{1}{|2n^2-3n-19|} \leq \frac{1}{n^2}$.
   We have $n \geq N \geq 7$, so $n - \frac{3}{2} \geq 5$. Thus, $(n - \frac{3}{2})^2 \geq 25 \geq \frac{85}{4}$. Expanding gives

$$n^2 - 3n + \frac{9}{4} \geq \frac{85}{4}.$$

   This gives $n^2 - 3n - 19 \geq 0$, and adding n^2 to both sides gives $2n^2 - 3n - 19 \geq n^2$.
   Since $n^2 > 0$, this shows $2n^2 - 3n - 19 > 0$, so $\frac{1}{|2n^2-3n-19|} = \frac{1}{2n^2-3n-19}$.
   Moreover, $2n^2 - 3n - 19 \geq n^2 > 0$ allows us to see that $0 < \frac{1}{2n^2-3n-19} \leq \frac{1}{n^2}$.
   In conclusion, we now know that $\frac{1}{|2n^2-3n-19|} \leq \frac{1}{n^2}$.
   Since $n \geq N \geq 7 > 1$ and $n \geq N \geq \frac{10}{\epsilon}$, we see that

$$\left|\frac{7n+3}{2n^2-3n-19}\right| \leq \frac{7n+3}{n^2} < \frac{7n+3n}{n^2} = \frac{10}{n} \leq \frac{10}{N} \leq \epsilon.$$

2. (a) Let $L \in \mathbb{R}$.
Let $\epsilon = 1$.
Let $N \in \mathbb{N}$.
Let $n = \max\{N, \lceil L \rceil + 1\}$.
We have $n \geq N$.
We also have $n \geq \lceil L \rceil + 1 \geq L + 1$, so $n - L \geq 1 \geq 0$ and

$$|n - L| = n - L \geq 1 = \epsilon.$$

(b) Let $L \in \mathbb{R}$.
Let $\epsilon = 1$.
Let $N \in \mathbb{N}$.
Let $n = 2\max\{N, \lceil L \rceil + 1\}$.
We have $n \geq \frac{n}{2} \geq N$.
We also have $n \geq \frac{n}{2} \geq \lceil L \rceil + 1 \geq L + 1$, so $n - L \geq 1 \geq 0$ and (since n is even)

$$|(-1)^n n - L| = |n - L| = n - L \geq 1 = \epsilon.$$

9 Using an assumed quantified sentence

Soon, we are going to want to use "(s_n) converges to L" as a hypothesis in a theorem statement. This means that we are going to have to assume the following sentence to be true.

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N \implies |s_n - L| < \epsilon.$$

The role of the quantifiers $\forall \epsilon > 0$ and $\exists N \in \mathbb{N}$ in our proofs will change drastically. We will be able to say things like "let $\epsilon = 1$" which were completely illegal when proving the sentence. And we will obtain N "for free" without having to say exactly what it is. The goal of this section is to avoid jumping in at the deep end and to provide intermediate examples, some of which you have already seen before.

9.1 Assuming a universally quantified statement

If you read section 3.7, then you have already seen some examples of *using* a universally quantified statement. The corollary of that section made use of the lemma immediately preceding it. The next lemma made use of the corollary. The main theorem, the triangle inequality, used the lemma right before it.

Using quantified sentences which only contain universal quantifiers is relatively easy. You have done it your whole life. For instance, you don't think twice about using

$$\forall x \in \mathbb{R},\ (x+1)^2 = x^2 + 2x + 1.$$

You just use it.

Let's see two more complicated examples.

Example 9.1.1. Let X, Y, and Z be sets. The following sentence is true, and it says "if f and g are injective, then $g \circ f$ is injective."

$$\begin{aligned}
&\forall f : X \longrightarrow Y,\ \forall g : Y \longrightarrow Z,\\
&\qquad \Bigg[\Big(\forall x_1 \in X,\ \forall x_2 \in X :\ f(x_1) = f(x_2) \implies x_1 = x_2\Big)\\
&\qquad\quad \text{and } \Big(\forall y_1 \in Y,\ \forall y_2 \in Y :\ g(y_1) = g(y_2) \implies y_1 = y_2\Big)\Bigg]\\
&\qquad \implies \Big(\forall x_1 \in X,\ \forall x_2 \in X :\ (g \circ f)(x_1) = (g \circ f)(x_2) \implies x_1 = x_2\Big).
\end{aligned}$$

Proof.

0. Let X, Y, and Z be sets.

1. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$.

2. We must verify the arrowed statement. Looking at the truth table for $\implies$, we see that we can break into two cases.

   (a) The premise is false. Then the arrowed statement is trivially true.

(b) The following sentence is true

$$\Big(\forall x_1 \in X,\ \forall x_2 \in X:\ f(x_1) = f(x_2) \implies x_1 = x_2\Big)$$
$$\text{and}\ \Big(\forall y_1 \in Y,\ \forall y_2 \in Y:\ g(y_1) = g(y_2) \implies y_1 = y_2\Big).$$

In this case, we must show that

$$\Big(\forall x_1 \in X,\ \forall x_2 \in X:\ (g \circ f)(x_1) = (g \circ f)(x_2) \implies x_1 = x_2\Big)$$

is true.

i. Let $x_1 \in X$.
ii. Let $x_2 \in X$.
iii. (Omitting trivial case.)
Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$. We must show that $x_1 = x_2$.
By definition of composition, we have $g(f(x_1)) = g(f(x_2))$.
Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so that $g(y_1) = g(y_2)$.
Because the second part of the "and" sentence is true, we know that

$$g(y_1) = g(y_2) \implies y_1 = y_2$$

is true. Thus, we conclude that $y_1 = y_2$, i.e. $f(x_1) = f(x_2)$.
Because the first part of the "and" sentence is true, we know that

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

is true. Thus, we conclude that $x_1 = x_2$.

□

Remark 9.1.2. This proof was more annoying to write than the one given in Part I, Homework 2, 3c. But doing things so systematically does allow us to make two extremely important observations.

First, in order to use

$$\forall y_1 \in Y,\ \forall y_2 \in Y:\ g(y_1) = g(y_2) \implies y_1 = y_2$$

we set $y_1 = f(x_1)$, $y_2 = f(x_2)$, the values that were relevant for us. This is completely the opposite of how we addressed universal quantifiers previously. The difference is because we're assuming this quantified sentence, rather than proving it.

We would have done the same again with x_1 and x_2 but the names of our variables already lined up conveniently with those in the quantified sentence. This leads us onto the second observation. The assumed quantified sentences do *not* declare or name variables. You should think of them as "dummy" variables. If I say "$\int_0^1 2x\ dx = 1$," you don't know what x is! Similarly, in our proof, x_1 and x_2 were not specific elements until we declared them in part 2bi and 2bii.

Example 9.1.3. Let X, Y, and Z be sets. The following sentence is true. It's 2a from the midterm.

$$\begin{aligned}&\forall f_1 : X \longrightarrow Y,\ \forall f_2 : X \longrightarrow Y,\ \forall g : Y \longrightarrow Z,\\ &\qquad\left[\left(\forall y_1 \in Y,\ \forall y_2 \in Y,\ g(y_1) = g(y_2) \implies y_1 = y_2\right) \text{ and } \left(g \circ f_1 = g \circ f_2\right)\right]\\ &\qquad\implies \left(\forall x \in X,\ f_1(x) = f_2(x)\right).\end{aligned}$$

I'll leave it to you to write the proof in the "format." It is a good exercise to help you understand the midterm question better.

Proof. Let X, Y, and Z be sets.

Let $f_1 : X \longrightarrow Y$, $f_2 : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be functions.

Suppose that

$$\left(\forall y_1 \in Y,\ \forall y_2 \in Y,\ g(y_1) = g(y_2) \implies y_1 = y_2\right) \text{ and } \left(g \circ f_1 = g \circ f_2\right)$$

is true.

Let $x \in X$.

We want to show that $f_1(x) = f_2(x)$.

Since $g \circ f_1 = g \circ f_2$, we have $(g \circ f_1)(x) = (g \circ f_2)(x)$. By definition of composition of functions, this is the same as $g(f_1(x)) = g(f_2(x))$.

Let $y_1 = f_1(x)$ and $y_2 = f_2(x)$, so that $g(y_1) = g(y_2)$.

Because the first clause of the "and" sentence is true, we know that

$$g(y_1) = g(y_2) \implies y_1 = y_2$$

is true. Thus, we conclude that $y_1 = y_2$, i.e. $f_1(x) = f_2(x)$. $\square$

Remark 9.1.4. Again, I prefer the way the solution to the midterm was written. But the quantifiers force you to declare variables in the correct order, so hopefully this way helps you understand previous errors, and learn more.

9.2 Assuming an existential

You already dealt with assuming an existential when you proved that for $n \in \mathbb{N}$, $\sim_n$ is an equivalence relation. Let's give some more examples before we discuss anything.

Example 9.2.1. The following sentence is true, and it encodes the English sentence "for all integers n_1 and n_2, if n_1 and n_2 are even, then $n_1 + n_2$ is even."

$$\begin{aligned}&\forall n_1 \in \mathbb{Z},\ \forall n_2 \in \mathbb{Z},\ \left[\left(\exists k_1 \in \mathbb{Z} :\ n_1 = 2k_1\right) \text{ and } \left(\exists k_2 \in \mathbb{Z} :\ n_2 = 2k_2\right)\right]\\ &\qquad\implies \left(\exists k \in \mathbb{Z} :\ n_1 + n_2 = 2k\right).\end{aligned}$$

Proof.

1. Let $n_1 \in \mathbb{Z}$.

2. Let $n_2 \in \mathbb{Z}$.

3. We must verify

$$\left[\left(\exists k_1 \in \mathbb{Z} :\ n_1 = 2k_1\right) \text{ and } \left(\exists k_2 \in \mathbb{Z} :\ n_2 = 2k_2\right)\right] \implies \left(\exists k \in \mathbb{Z} :\ n_1 + n_2 = 2k\right).$$

   Looking at the truth table for $\implies$, we see that we can break into two cases.

   (a) $\left(\exists k_1 \in \mathbb{Z} :\ n_1 = 2k_1\right)$ and $\left(\exists k_2 \in \mathbb{Z} :\ n_2 = 2k_2\right)$ is false.
   Then the arrowed statement is trivially true.

   (b) The following sentence is true.

$$\left(\exists k_1 \in \mathbb{Z} :\ n_1 = 2k_1\right) \text{ and } \left(\exists k_2 \in \mathbb{Z} :\ n_2 = 2k_2\right) \tag{9.2.2}$$

   In this case, we must show that $\left(\exists k \in \mathbb{Z} :\ n_1 + n_2 = 2k\right)$ is true.

   i. Let $k_1, k_2 \in \mathbb{Z}$ be chosen so that $n_1 = 2k_1$ and $n_2 = 2k_2$. We can choose such k_1 and k_2 because (9.2.2) is true. Let $k = k_1 + k_2$. We certainly have $k \in \mathbb{Z}$.

   ii. We see that $n_1 + n_2 = 2k_1 + 2k_2 = 2(k_1 + k_2) = 2k$.

□

Example 9.2.3. Let X be a non-empty set. The following sentence is true.

$$\forall f : X \longrightarrow \mathbb{R},\ \left(\exists x' \in X :\ f(x') = 2\pi\right) \implies \left(\exists x \in X :\ \sin(f(x)) = 0\right).$$

Proof.

0. Let X be a non-empty set.

1. Let $f : X \longrightarrow \mathbb{R}$ be a function.

2. We must verify

$$\left(\exists x' \in X :\ f(x') = 2\pi\right) \implies \left(\exists x \in X :\ \sin(f(x)) = 0\right).$$

   Looking at the truth table for $\implies$, we see that we can break into two cases.

   (a) $\left(\exists x' \in X :\ f(x') = 2\pi\right)$ is false.
   Then the arrowed statement is trivially true.

(b) The following sentence is true.

$$\left(\exists x' \in X : \ f(x') = 2\pi\right) \tag{9.2.4}$$

In this case, we must show that $\left(\exists x \in X : \ \sin(f(x)) = 0\right)$ is true.

i. Let $x' \in X$ be chosen so that $f(x') = 2\pi$. We can choose such an x' because (9.2.4) is true. Let $x = x'$. We certainly have $x \in X$.

ii. We see that $\sin(f(x)) = \sin(f(x')) = \sin(2\pi) = 0$.

$\square$

Example 9.2.5. The following sentence is true.

$$\forall a_0 \in \mathbb{R}, \ \forall a_1 \in \mathbb{R}, \ \forall a_2 \in \mathbb{R}, \ \left[\exists x \in \mathbb{R} : \ a_0 + a_1 x + a_2 x^2 + x^3 = 0\right] \implies \left[\exists y \in \mathbb{R} : \ a_0 + 2a_1 y + 4a_2 y^2 + 8y^3 = 0\right].$$

Proof.

1. Let $a_0, a_1, a_2 \in \mathbb{R}$.

2. We must verify

$$\left[\exists x \in \mathbb{R} : \ a_0 + a_1 x + a_2 x^2 + x^3 = 0\right] \implies \left[\exists y \in \mathbb{R} : \ a_0 + 2a_1 y + 4a_2 y^2 + 8y^3 = 0\right].$$

Looking at the truth table for $\implies$, we see that we can break into two cases.

(a) $\left[\exists x \in \mathbb{R} : \ a_0 + a_1 x + a_2 x^2 + x^3 = 0\right]$ is false.
Then the arrowed statement is trivially true.

(b) The following sentence is true.

$$\left[\exists x \in \mathbb{R} : \ a_0 + a_1 x + a_2 x^2 + x^3 = 0\right] \tag{9.2.6}$$

In this case, we must show that $\left[\exists y \in \mathbb{R} : \ a_0 + 2a_1 y + 4a_2 y^2 + 8y^3 = 0\right]$ is true.

i. Let $x \in \mathbb{R}$ be chosen so that $a_0 + a_1 x + a_2 x^2 + x^3 = 0$. We can choose such an x because (9.2.6) is true. Let $y = \frac{x}{2}$. We certainly have $y \in \mathbb{R}$.

ii. We see that

$$a_0 + 2a_1 y + 4a_2 y^2 + 8y^3 = a_0 + 2a_1 \cdot \frac{x}{2} + 4a_2 \cdot \frac{x^2}{2^2} + 8 \cdot \frac{x^3}{2^3} = a_0 + a_1 x + a_2 x^2 + x^3 = 0.$$

$\square$

Remark 9.2.7. The main thing to notice about these proofs is the difference between:

- asserting that an element with a property exists.
- actually picking an element with the given property.

Normally in analysis proofs, I will only say that an element with some property exists if I am going to pick such an element, so why should this issue matter? In analysis, many of the definitions are long quantified sentences. It can be helpful for students to write down what they are assuming to be true. Perhaps they want to record that a sequence (s_n) converges to a real number L. So they write down, "We know

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N \implies |s_n - L| < \epsilon."$$

Does this sentence give an $N \in \mathbb{N}$? Definitely not!

It becomes even more confusing once we know more than one thing at a time, because variables can get used twice. We don't even have to consider a sentence as complicated as the "convergence" sentence to see this being a problem. I might observe that

$$\exists x \in \mathbb{R} :\ x > 0 \text{ and } x^2 = 2$$

is true. Then a minute later, I might observe that

$$\exists x \in \mathbb{R} :\ x < 0 \text{ and } x^2 = 2$$

is true. If I now start saying "x this" and "x that," am I talking about $\sqrt{2}$ or $-\sqrt{2}$?! This is very different to the following paragraph.

Notice that $(\exists x \in \mathbb{R} :\ x > 0 \text{ and } x^2 = 2)$ is true and notice that $(\exists x \in \mathbb{R} :\ x < 0 \text{ and } x^2 = 2)$ is true. Let $y \in \mathbb{R}$ be chosen so that $y > 0$ and $y^2 = 2$. We even had the opportunity to use a different variable name this way. From the moment I declared it, y is fixed!

9.3 Exercises

1. I delayed this question until now because part (b) requires assuming an existential.

   (a) Prove the following sentence is true.

   $$\forall a \in \mathbb{R} \setminus \{0\},\ \forall b \in \mathbb{R},\ \forall c \in \mathbb{R},\ \left(b^2 - 4ac \geq 0 \implies \left(\exists x \in \mathbb{R} :\ ax^2 + bx + c = 0\right)\right).$$

   (b) Prove the following sentence is true.

   $$\forall a \in \mathbb{R} \setminus \{0\},\ \forall b \in \mathbb{R},\ \forall c \in \mathbb{R},\ \left(\left(\exists x \in \mathbb{R} :\ ax^2 + bx + c = 0\right) \implies b^2 - 4ac \geq 0\right).$$

   (c) The following quantified sentence is true, although you probably won't have a good proof of it just yet.

   $$\forall a_0 \in \mathbb{R},\ \forall a_1 \in \mathbb{R},\ \forall a_2 \in \mathbb{R},\ \forall a_3 \in \mathbb{R} \setminus \{0\},\ \exists x \in \mathbb{R} :\ a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0.$$

   Can you express the meaning of this sentence in words? I can say it in 3 words.

9.4 Solutions to previous exercises

1. (a) Let $a \in \mathbb{R} \setminus \{0\}$.
   Let $b \in \mathbb{R}$.
   Let $c \in \mathbb{R}$.
   Suppose $b^2 - 4ac \geq 0$. We wish to show $(\exists x \in \mathbb{R} :\ ax^2 + bx + c = 0)$.

   i. Let $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$.
   Since $a \neq 0$ and $b^2 - 4ac \geq 0$, x is a well-defined real number.

   ii. It is basic algebra to check that $ax^2 + bx + c = 0$.

   Here's the algebra, though I'm not thinking of this as part of the proof.

   $$\begin{aligned} x &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ 2ax + b &= \sqrt{b^2 - 4ac} \\ (2ax + b)^2 &= b^2 - 4ac \\ 4a^2x^2 + 4abx + b^2 &= b^2 - 4ac \\ 4a(ax^2 + bx + c) &= 0 \\ ax^2 + bx + c &= 0 \end{aligned}$$

   (b) Let $a \in \mathbb{R} \setminus \{0\}$.
   Let $b \in \mathbb{R}$.
   Let $c \in \mathbb{R}$.
   Suppose $(\exists x \in \mathbb{R} :\ ax^2 + bx + c = 0)$. We wish to show $b^2 - 4ac \geq 0$.
   Pick an $x \in \mathbb{R}$ such that $ax^2 + bx + c = 0$.
   Then we see that

   $$0 \leq (2ax + b)^2 = 4a^2x^2 + 4abx + b^2 = 4a(ax^2 + bx) + b^2 = 4a(-c) + b^2 = b^2 - 4ac.$$

   (c) Cubics have roots.

9.5 Assuming a more complicated quantified sentence

Example 9.5.1. The following sentence is true.

$$\forall f : \mathbb{R} \longrightarrow \mathbb{R}, \ \left[\forall \epsilon' > 0, \ \exists q' \in \mathbb{Q} : \ |f(q')| < \epsilon'\right] \implies \left[\forall \epsilon > 0, \ \exists q \in \mathbb{Q} : \ |f(3q)| < \frac{\epsilon}{2}\right]$$

Remark 9.5.2. I'm tired of typing out the trivial case of an arrowed statement. Hopefully, writing it many times has helped you understand the precise meaning of $\implies$.

Proof.

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function.

2. Suppose that the following sentence is true.

$$\forall \epsilon' > 0, \ \exists q' \in \mathbb{Q} : \ |f(q')| < \epsilon' \tag{9.5.3}$$

   We must verify ($\forall \epsilon > 0, \ \exists q \in \mathbb{Q} : \ |f(3q)| < \frac{\epsilon}{2}$).

   (a) Let $\epsilon > 0$.

   (b) Let $\epsilon' = \frac{\epsilon}{2}$. Then $\epsilon' > 0$.
   Using that (9.5.3) is true, pick a $q' \in \mathbb{Q}$ such that

$$|f(q')| < \epsilon'.$$

   Let $q = \frac{q'}{3}$. Then $q \in \mathbb{Q}$.

   (c) We have $|f(3q)| = |f(q')| < \epsilon' = \frac{\epsilon}{2}$.

□

Example 9.5.4. Let X, Y, and Z be sets. The following sentence is true, and it says "if f and g are surjective, then $g \circ f$ is surjective."

$$\begin{aligned} &\forall f : X \longrightarrow Y, \ \forall g : Y \longrightarrow Z, \\ &\qquad \left[\left(\forall y_1 \in Y, \ \exists x' \in X : \ f(x') = y_1\right) \text{ and } \left(\forall z' \in Z, \ \exists y_2 \in Y : \ g(y_2) = z'\right)\right] \\ &\qquad \implies \left(\forall z \in Z, \ \exists x \in X : \ (g \circ f)(x) = z\right). \end{aligned}$$

Proof.

0. Let X, Y, and Z be sets.

1. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be functions.

2. Suppose that the following sentences are true.

$$\forall y_1 \in Y, \ \exists x' \in X : \ f(x') = y_1 \tag{9.5.5}$$

$$\forall z' \in Z, \ \exists y_2 \in Y : \ g(y_2) = z' \tag{9.5.6}$$

   We must verify ($\forall z \in Z, \ \exists x \in X : \ (g \circ f)(x) = z$).

(a) Let $z \in Z$.

(b) Let $z' = z$. Using that (9.5.6) is true, pick a $y_2 \in Y$ such that

$$g(y_2) = z'. \tag{9.5.7}$$

Let $y_1 = y_2$. Using that (9.5.5) is true, pick an $x' \in X$ such that

$$f(x') = y_1. \tag{9.5.8}$$

Let $x = x'$.

(c) We have $(g \circ f)(x) = g(f(x')) = g(y_1) = g(y_2) = z' = z$.
The first equality uses the definition of composition of functions and $x = x'$. The second uses (9.5.8). The third uses $y_1 = y_2$. The fourth uses (9.5.7). The last uses $z' = z$.

$\square$

Example 9.5.9. Let X, Y and Z be sets. The following sentence is true. It's 2b from the midterm.

$$\begin{aligned}&\forall f : X \longrightarrow Y,\ \forall g_1 : Y \longrightarrow Z,\ \forall g_2 : Y \longrightarrow Z,\\ &\qquad\left[\left(\forall y' \in Y,\ \exists x \in X:\ f(x) = y'\right) \text{ and } \left(g_1 \circ f = g_2 \circ f\right)\right]\\ &\qquad\implies \left(\forall y \in Y,\ g_1(y) = g_2(y)\right).\end{aligned}$$

Proof. Let X, Y, and Z be sets.

Let $f : X \longrightarrow Y$, $g_1 : Y \longrightarrow Z$, $g_2 : Y \longrightarrow Z$ be functions.

Suppose that

$$\left(\forall y' \in Y,\ \exists x \in X:\ f(x) = y'\right) \text{ and } \left(g_1 \circ f = g_2 \circ f\right)$$

is true.

Let $y \in Y$.

We want to show that $g_1(y) = g_2(y)$.

Let $y' = y$.

Using that the first part of the "and" sentence is true, pick an $x \in X$ such that $f(x) = y'$.

Since $g_1 \circ f = g_2 \circ f$, we have $(g_1 \circ f)(x) = (g_2 \circ f)(x)$. By definition of composition of functions, this is the same as $g_1(f(x)) = g_2(f(x))$. Thus,

$$g_1(y) = g_1(y') = g_1(f(x)) = g_2(f(x)) = g_2(y') = g_2(y).$$

$\square$

Remark 9.5.10. All the previous proofs were purposefully written very algorithmically.

I made sure to use primes and subscripts so that no variables were used twice in the sentences that we were proving. Sometimes this will not be the case and you may want to add these yourself to avoid confusion and to help you keep your order of declarations straight.

I may write more here, but it is probably more useful for me to talk with you when I see your proofs. Almost every student struggles with this...

9.6 Exercises

1. Prove that the following sentence is true.

$$\forall \text{ sequences } (s_n)_{n=1}^{\infty},\ \forall L \in \mathbb{R},\ \Big(\forall \epsilon_1 > 0,\ \exists N_1 \in \mathbb{N}:\ \forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - L| < \epsilon_1\Big)$$
$$\iff \Big(\forall \epsilon_2 > 0,\ \exists N_2 \in \mathbb{R}:\ \forall n \in \mathbb{N},\ n > N_2 \implies |s_n - L| < \epsilon_2\Big).$$

Solution...

Let $(s_n)_{n=1}^{\infty}$ be a sequence.

Let $L \in \mathbb{R}$.

First, suppose

$$\forall \epsilon_1 > 0,\ \exists N_1 \in \mathbb{N}:\ \forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - L| < \epsilon_1. \tag{9.6.1}$$

We will verify

$$\forall \epsilon_2 > 0,\ \exists N_2 \in \mathbb{R}:\ \forall n \in \mathbb{N},\ n > N_2 \implies |s_n - L| < \epsilon_2.$$

(a) Let $\epsilon_2 > 0$.

(b) Let $\epsilon_1 = \epsilon_2$. Using (9.6.1), pick $N_1 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - L| < \epsilon_1. \tag{9.6.2}$$

Let $N_2 = N_1$. Since $\mathbb{N} \subseteq \mathbb{R}$, we have $N_2 \in \mathbb{R}$.

(c) Let $n \in \mathbb{N}$.

(d) Suppose $n > N_2$. We wish to show $|s_n - L| < \epsilon_2$.
We have $n > N_2 = N_1$. In particular, $n \geq N_1$. Thus, (9.6.2) gives $|s_n - L| < \epsilon_1$.
Since $\epsilon_1 = \epsilon_2$, we have $|s_n - L| < \epsilon_2$.

Next, suppose

$$\forall \epsilon_2 > 0,\ \exists N_2 \in \mathbb{R}:\ \forall n \in \mathbb{N},\ n > N_2 \implies |s_n - L| < \epsilon_2 \tag{9.6.3}$$

We will verify

$$\forall \epsilon_1 > 0,\ \exists N_1 \in \mathbb{N}:\ \forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - L| < \epsilon_1.$$

(a) Let $\epsilon_1 > 0$.

(b) Let $\epsilon_2 = \epsilon_1$. Using (9.6.3), pick $N_2 \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N},\ n > N_2 \implies |s_n - L| < \epsilon_2. \tag{9.6.4}$$

Let $N_1 = \lceil |N_2| \rceil + 1$. Then $N_1 \in \mathbb{N}$.

(c) Let $n \in \mathbb{N}$.

(d) Suppose $n \geq N_1$. We wish to show $|s_n - L| < \epsilon_1$.
We have $n \geq N_1 = \lceil |N_2| \rceil + 1 \geq N_2 + 1 > N_2$. Thus, (9.6.4) gives $|s_n - L| < \epsilon_2$.
Since $\epsilon_2 = \epsilon_1$, we have $|s_n - L| < \epsilon_1$.

10 More on sequences

10.1 The algebra of limits

It is a little tedious to verify the definition of a convergent sequence every time we want to show that a sequence converges. In mathematics, we often prove theorems to save ourselves from having to do similar work over and over again. That is the purpose of the algebra of limits. It says things like, "the sum of two convergent sequences is a convergent sequence."

The first result in the algebra of limits which we will prove is the following.

Theorem 10.1.1. *Suppose $(s_n)_{n=1}^{\infty}$ is a sequence and $s, c \in \mathbb{R}$. If (s_n) converges to s, then $(cs_n)_{n=1}^{\infty}$ converges to cs. We could also state this theorem by saying that the following sentence is true.*

$$\forall \textit{sequences } (s_n)_{n=1}^{\infty},\ \forall s \in \mathbb{R},\ \forall c \in \mathbb{R},\ \lim_{n\to\infty} s_n = s \implies \lim_{n\to\infty} cs_n = cs.$$

Proof attempt.

1. Let (s_n) be a sequence.
2. Let $s \in \mathbb{R}$.
3. Let $c \in \mathbb{R}$.
4. We must verify $(\lim_{n\to\infty} s_n = s \implies \lim_{n\to\infty} cs_n = cs)$.
   (a) Case 1: $\lim_{n\to\infty} s_n = s$ is false. Trivial.
   (b) Case 2: $\lim_{n\to\infty} s_n = s$ is true. We must show that $\lim_{n\to\infty} cs_n = cs$ is true. Both of these expressions have a definition associated with them. So we can expand on this and say the following.
      - The following sentence is true.
        $$\forall \epsilon' > 0,\ \exists N' \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N' \implies |s_n - s| < \epsilon'.$$
        (The use of ϵ' and N' will become clear in the next subsection.)
      - We must show that the following sentence is true.
        $$\forall \epsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N \implies |cs_n - cs| < \epsilon.$$

      We verify the last statement.
      i. Let $\epsilon > 0$
      ii. Let $N =$ BLAH.
      iii. Let $n \in \mathbb{N}$.
      iv. We must verify $(n \geq N \implies |cs_n - cs| < \epsilon)$.
         A. Case 1: $n < N$. Trivial.
         B. Case 2: $n \geq N$. In this case
         $$|cs_n - cs| = |c||s_n - s| \ldots \text{BLAH}.$$

□

We have to fill in the BLAHs. How can we possibly write down an N if we have no idea what (cs_n) looks like?

10.2 The game and the computer program, finishing the proof of 10.1.1

In earlier subsections we verified that some sequences converge. After figuring out the limit L of a sequence, this verification comes down to playing an "ϵ-N game." The goal of the ϵ-N game is to figure out N in terms of ϵ, and it must be an $N \in \mathbb{N}$ making the relevant statement

$$\forall n \in \mathbb{N},\ n \geq N \implies |s_n - s| < \epsilon \tag{10.2.1}$$

true. When we wrote our proofs, we wrote them in the style of a computer program.

The information that a sequence converges to a limit can be viewed as giving us such a computer program. If we give the computer program a postive real number ϵ', it will return a natural number N' making the relevant statement $(10.2.1)'$ true.

The key to completing the proof of theorem 10.1.1 is to use such a "computer program" to specify an N.

Completing the proof of theorem 10.1.1.

4. (b) i. Let $\epsilon > 0$.
   ii. This is where the bulk of the argument happens.
      A. Let $\epsilon' = \frac{\epsilon}{1+|c|}$. Then $\epsilon' > 0$.
      B. Since
      $$\forall \epsilon' > 0,\ \exists N' \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N' \implies |s_n - s| < \epsilon'$$
      is true, we obtain an $N' \in \mathbb{N}$ such that the following sentence is true.
      $$\forall n \in \mathbb{N},\ n \geq N' \implies |s_n - s| < \epsilon'.$$
      C. Let $N = N'$.
   iii. Let $n \in \mathbb{N}$.
   iv. We must verify $(n \geq N \implies |cs_n - cs| < \epsilon)$.
      A. Case 1: $n < N$. Trivial.
      B. Case 2: $n \geq N$.
      Since $N = N'$, we have $n \geq N'$, and so $|s_n - s| < \epsilon'$. Thus,
      $$|cs_n - cs| = |c||s_n - s| \leq |c|\epsilon' = |c|\frac{\epsilon}{1+|c|} < \epsilon.$$

□

Notice how algorithmically this proof is written.

- We say what ϵ' is in terms of c and ϵ which were already declared in 2. and 4.(b)i., respectively.
- We obtain N' from the "computer program" and ϵ' which were already declared in 4.(b)• and 4.(b)ii.A., respectively.
- We say what N is in terms of N' which was declared in the step before.
- Finally, we check that everything works as it is supposed to.

10.3 Another algebra of limits proof, more game/computer program

Let's see another similar example.

Theorem 10.3.1. *Suppose $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are sequences, and that $s, t \in \mathbb{R}$. If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)_{n=1}^{\infty}$ converges to $s + t$.*

We could also state this theorem by saying that the following sentence is true.

$$\forall \text{sequences } (s_n)_{n=1}^{\infty},\ \forall \text{sequences } (t_n)_{n=1}^{\infty},\ \forall s \in \mathbb{R},\ \forall t \in \mathbb{R},$$
$$\left(\lim_{n\to\infty} s_n = s \text{ and } \lim_{n\to\infty} t_n = t\right) \implies \lim_{n\to\infty} (s_n + t_n) = s + t.$$

Proof.

1. Let (s_n) be a sequence.
2. Let (t_n) be a sequence.
3. Let $s \in \mathbb{R}$.
4. Let $t \in \mathbb{R}$.
5. We must verify

$$\left(\lim_{n\to\infty} s_n = s \text{ and } \lim_{n\to\infty} t_n = t\right) \implies \lim_{n\to\infty} (s_n + t_n) = s + t.$$

   (a) Case 1: ($\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$) is false. Trivial.
   (b) Case 2: ($\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$) is true.
   We must show that $\lim_{n\to\infty}(s_n + t_n) = s + t$ is true. Since all of these expressions have a definition associated with them, we can expand on this and say the following.

   - The following sentence is true.

$$\forall \epsilon_1 > 0,\ \exists N_1 \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - s| < \epsilon_1.$$

   - The following sentence is true.

$$\forall \epsilon_2 > 0,\ \exists N_2 \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N_2 \implies |t_n - t| < \epsilon_2.$$

   - We must show that the following sentence is true.

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N \implies |(s_n + t_n) - (s + t)| < \epsilon.$$

   We verify the last statement.

   i. Let $\epsilon > 0$.
   ii. This is where the bulk of the argument happens.
      A. Let $\epsilon_1 = \frac{\epsilon}{2}$. Then $\epsilon_1 > 0$.
      B. Let $\epsilon_2 = \frac{\epsilon}{2}$. Then $\epsilon_2 > 0$.

C. Since

$$\forall \epsilon_1 > 0,\ \exists N_1 \in \mathbb{N}:\ \forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - s| < \epsilon_1.$$

is true, we obtain an $N_1 \in \mathbb{N}$ such that the following sentence is true.

$$\forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - s| < \epsilon_1.$$

D. Since

$$\forall \epsilon_2 > 0,\ \exists N_2 \in \mathbb{N}:\ \forall n \in \mathbb{N},\ n \geq N_2 \implies |t_n - t| < \epsilon_2.$$

is true, we obtain an $N_2 \in \mathbb{N}$ such that the following sentence is true.

$$\forall n \in \mathbb{N},\ n \geq N_2 \implies |t_n - t| < \epsilon_2.$$

E. Let $N = \max\{N_1, N_2\}$.

iii. Let $n \in \mathbb{N}$.

iv. We must verify $(n \geq N \implies |(s_n + t_n) - (s + t)| < \epsilon)$.

A. Case 1: $n < N$. Trivial.

B. Case 2: $n \geq N$.
Since $N = \max\{N_1, N_2\} \geq N_1$, we have $n \geq N_1$, and so $|s_n - s| < \epsilon_1$.
Since $N = \max\{N_1, N_2\} \geq N_2$, we have $n \geq N_2$, and so $|t_n - t| < \epsilon_2$.
Thus,

$$|(s_n + t_n) - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \epsilon_1 + \epsilon_2 = \epsilon.$$

$\square$

Again notice how algorithmically this proof is written. Here is how you would see most mathematicians write such a proof.

The proof of theorem 10.3.1 written differently. Suppose (s_n) and (t_n) are sequences, that $s, t \in \mathbb{R}$, that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$.

We wish to show that $\lim_{n\to\infty}(s_n + t_n) = s + t$, so let $\epsilon > 0$.

Since (s_n) converges to s, there exists an $N_1 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - s| < \frac{\epsilon}{2}.$$

Since (t_n) converges to t, there exists an $N_2 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ n \geq N_2 \implies |t_n - t| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$, and suppose that $n \in \mathbb{N}$ and $n \geq N$. Then

$$|(s_n + t_n) - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

You can see that this proof does exactly what the previous proof does. However, it misses out certain trivialities. It also misses out explaining exactly what quantified sentence is being checked at what time, and is less explicit about how known-to-be-true quantified sentences are being used. The order of declarations is still *exactly* the same.

Eventually, I'll let you write proofs like the one just written. The reason I won't just yet is that the way I've been doing it forces you to declare variables in the correct order. Thus, mistakes you make should be elsewhere, and we can discuss them as opposed to quantifier errors.

Remark 10.3.2. Let me call your attention to something about the way I wrote the previous algebra of limits proofs. In the first proof, I wrote...

Since

$$\forall \epsilon' > 0,\ \exists N' \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N' \implies |s_n - s| < \epsilon'$$

is true, we obtain an $N' \in \mathbb{N}$ such that the following sentence is true.

$$\forall n \in \mathbb{N},\ n \geq N' \implies |s_n - s| < \epsilon'.$$

When I write

$$\forall \epsilon' > 0,\ \exists N' \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N' \implies |s_n - s| < \epsilon',$$

I write it as a formula, and remark that the whole statement is true. Then I go ahead and actually use the truth of this statement (ϵ' was already specified). I write "we obtain an $N' \in \mathbb{N}$" in the bulk of the text, in *words* as opposed to quantifiers, to emphasize that this is an N' that I am actually going to use. I follow this up by stating the property that this N' has (as a formula):

$$\forall n \in \mathbb{N},\ n \geq N' \implies |s_n - s| < \epsilon'.$$

These are subtle uses of text/formulae. They are probably "me-teaching-95-things" but there is method in the madness, and it does emphasize the different things which going on. If you write this way, you will be clearer than most mathematicians.

10.4 Exercises

Definition. We say that a sequence $(s_n)_{n=1}^{\infty}$ is bounded **iff** the following sentence is true.

$$\exists M > 0 : \ \forall n \in \mathbb{N}, \ |s_n| \leq M.$$

1. Prove that the following sentence is true.

$$\forall \text{sequences } (s_n)_{n=1}^{\infty}, \ \forall \text{sequences } (t_n)_{n=1}^{\infty},$$
$$\left(\lim_{n \to \infty} s_n = 0 \text{ and } (t_n) \text{ is bounded} \right) \implies \lim_{n \to \infty} s_n t_n = 0.$$

2. Suppose $s, t \in \mathbb{R}$ and that $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are sequences converging to s and t, respectively. Show that $(s_n t_n)_{n=1}^{\infty}$ converges to st.

   You should use the fact that for all $n \in \mathbb{N}$, $s_n t_n = (s_n - s)t_n + s(t_n - t) + st$, the following theorem, the previous question, and the algebra of limits results proved in these notes.

   Theorem. *Suppose $(s_n)_{n=1}^{\infty}$ converges. Then (s_n) is bounded.*

   Definition. The *contrapositive* of $(P \implies Q)$ is $(\text{NOT}(Q) \implies \text{NOT}(P))$. This has the same truth table as $(P \implies Q)$.

3. Consider the following statement.

$$\forall x \in \mathbb{R}, \ \left(\forall \epsilon > 0, \ |x| < \epsilon \right) \implies x = 0.$$

   It's true, but it is basically impossible to say something useful in the proof without using the contrapositive or a proof by contradiction.

   (a) Give the contrapositive of the statement above. Leave $\forall x \in \mathbb{R}$ alone; just focus on the arrowed statement.

   (b) Verify the sentence you get as your answer to (a).

   (c) Notice this gives strange strategy for showing that two real numbers are equal:

$$\forall x \in \mathbb{R}, \ \forall y \in \mathbb{R}, \ \left(\forall \epsilon > 0, \ |x - y| < \epsilon \right) \implies x = y.$$

4. Prove the following statement.

$$\forall \text{sequences } (s_n)_{n=1}^{\infty}, \ \forall L_1 \in \mathbb{R}, \ \forall L_2 \in \mathbb{R}$$
$$\Bigg(\left(\forall \epsilon > 0, \ \exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N \implies |s_n - L_1| < \epsilon \right) \text{ and }$$
$$\left(\forall \epsilon > 0, \ \exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N \implies |s_n - L_2| < \epsilon \right) \Bigg) \implies \left(\forall \epsilon > 0, \ |L_1 - L_2| < \epsilon \right).$$

   What have you proved?! Remember 3c!

5. Suppose that $t \in \mathbb{R} \setminus \{0\}$ and that $(t_n)_{n=1}^{\infty}$ is a sequence converging to t.

   (a) Prove the following sentence.

   $$\exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N \implies t_n \neq 0.$$

   (b) For simplicity, assume the stronger statement that

   $$\forall n \in \mathbb{N}, \ t_n \neq 0.$$

   Prove that $\left(\frac{1}{t_n}\right)_{n=1}^{\infty}$ converges to $\frac{1}{t}$.

6. (a) Prove the following sentence is true.

   $$\forall L \in \mathbb{R}, \ \forall a \in \mathbb{R}, \ L < a \implies \Big(\exists \epsilon > 0 : \ \forall s \in \mathbb{R}, \ |s - L| < \epsilon \implies s < a\Big).$$

   (b) Prove that the following sentence is true.

   $$\begin{aligned}&\forall \text{sequences } (s_n)_{n=1}^{\infty}, \ \forall L \in \mathbb{R}, \ \forall a \in \mathbb{R},\\ &\Big(\lim_{n\to\infty} s_n = L \text{ and } L < a\Big) \implies \Big(\exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N \implies s_n < a\Big).\end{aligned}$$

   You should definitely be using part (a).

   (c) The contrapositive of part (b) tells you that the following sentence is true.

   $$\begin{aligned}&\forall \text{sequences } (s_n)_{n=1}^{\infty}, \ \forall L \in \mathbb{R}, \ \forall a \in \mathbb{R},\\ &\Big(\forall N \in \mathbb{N}, \ \exists n \in \mathbb{N} : \ n \geq N \text{ and } s_n \geq a\Big) \implies \Big(\text{NOT}\big(\lim_{n\to\infty} s_n = L\big) \text{ or } L \geq a\Big).\end{aligned}$$

   Use this to prove that the following sentence is true.

   $$\begin{aligned}&\forall \text{sequences } (s_n)_{n=1}^{\infty}, \ \forall L \in \mathbb{R}, \ \forall a \in \mathbb{R},\\ &\Big(\lim_{n\to\infty} s_n = L \ \text{ and } \ \forall n \in \mathbb{N}, \ s_n \geq a\Big) \implies L \geq a.\end{aligned}$$

   This proof might be a good time to try out using the words "in particular."

   (d) I just guided you through a proof by giving you all necessary ideas one by one. However, you can write a far more concise proof after doing away with the "format." Here's the question I set the first time I taught 131A...

   Suppose $(s_n)_{n=1}^{\infty}$ is a sequence, $L \in \mathbb{R}$, $a \in \mathbb{R}$, that (s_n) converges to L, and that for all $n \in \mathbb{N}$, $s_n \geq a$. Prove that $L \geq a$. Start your proof by writing "suppose for contradiction that $L < a$."

7. Prove the theorem mentioned above, that convergent sequences are bounded.

10.5 Solutions to some exercises

1. Let (s_n) be a sequence.

   Let (t_n) be a sequence.

   Suppose (s_n) converges to 0 and (t_n) is bounded, i.e. the following sentences are true.

$$\forall \epsilon' > 0,\ \exists N' \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N' \implies |s_n - 0| < \epsilon'. \tag{10.5.1}$$

$$\exists M > 0 :\ \forall n \in \mathbb{N},\ |t_n| \leq M. \tag{10.5.2}$$

   We wish to show that $(s_n t_n)$ converges to 0, i.e.

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N \implies |s_n t_n - 0| < \epsilon.$$

   (a) Let $\epsilon > 0$.

   (b) Using (10.5.2), pick $M > 0$ with the property that

$$\forall n \in \mathbb{N},\ |t_n| \leq M. \tag{10.5.3}$$

   Let $\epsilon' = \frac{\epsilon}{M}$. Then $\epsilon' > 0$. Using (10.5.1), pick $N' \in \mathbb{N}$ with the property that

$$\forall n \in \mathbb{N},\ n \geq N' \implies |s_n - 0| < \epsilon'. \tag{10.5.4}$$

   Let $N = N'$.

   (c) Let $n \in \mathbb{N}$.

   (d) Suppose $n \geq N$. We wish to show that $|s_n t_n - 0| < \epsilon$.
   By (10.5.3), we have $|t_n| \leq M$.
   We have $n \geq N = N'$, and so, by (10.5.4), we have $|s_n - 0| \leq \epsilon'$. Thus,

$$|s_n t_n - 0| = |s_n t_n| \leq |s_n| \cdot M = |s_n - 0| \cdot M < \epsilon' \cdot M = \epsilon.$$

2. Suppose $s, t \in \mathbb{R}$ and that $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are sequences converging to s and t, respectively. We want to show that $(s_n t_n)_{n=1}^{\infty}$ converges to st.

   First, note that the constant sequences $(-s)_{n=1}^{\infty}$ and $(-t)_{n=1}^{\infty}$ converge to $-s$ and $-t$, respectively. The proof of both of these facts begins with "Let $\epsilon > 0$, let $N = 1, \ldots$" By theorem 10.3.1, we see that $(s_n - s)_{n=1}^{\infty}$ and $(t_n - t)_{n=1}^{\infty}$ both converge to 0.

   The theorem quoted after the question tells us that $(t_n)_{n=1}^{\infty}$ is bounded, and so the previous question applies to say $((s_n - s)t_n)_{n=1}^{\infty}$ converges to 0.

   The constant sequence $(s)_{n=1}^{\infty}$ is bounded (let $M = |s| + 1$), so the previous question applies to say $(s(t_n - t))_{n=1}^{\infty}$ converges to 0.

   By theorem 10.3.1, we see that $((s_n - s)t_n + s(t_n - t))_{n=1}^{\infty}$ converges to 0.

   The constant sequence $(st)_{n=1}^{\infty}$ converges to st. By theorem 10.3.1, we see that

$$(s_n t_n)_{n=1}^{\infty} = ((s_n - s)t_n + s(t_n - t) + st)_{n=1}^{\infty}$$

   converges to st.

3. (a) $\forall x \in \mathbb{R},\ x \neq 0 \implies \left(\exists \epsilon > 0 :\ |x| \geq \epsilon\right).$

   (b) Let $x \in \mathbb{R}$.
   Suppose $x \neq 0$.
   Let $\epsilon = |x|$. Then $\epsilon > 0$.
   Moreover, $|x| = \epsilon$. In particular $|x| \geq \epsilon$.

   (c) Cool.

4. Let (s_n) be a sequence. Let $L_1 \in \mathbb{R}$. Let $L_2 \in \mathbb{R}$.

   Suppose that the following sentences are true.

$$\forall \epsilon_1 > 0,\ \exists N_1 \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - L_1| < \epsilon_1 \tag{10.5.5}$$

$$\forall \epsilon_2 > 0,\ \exists N_2 \in \mathbb{N} :\ \forall n \in \mathbb{N},\ n \geq N_2 \implies |s_n - L_2| < \epsilon_2 \tag{10.5.6}$$

   We will verify $\left(\forall \epsilon > 0,\ |L_1 - L_2| < \epsilon\right)$.

   (a) Let $\epsilon > 0$.

   (b) Let $\epsilon_1 = \frac{\epsilon}{2}$. Use (10.5.5) to pick $N_1 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ n \geq N_1 \implies |s_n - L_1| < \epsilon_1.$$

   Let $\epsilon_2 = \frac{\epsilon}{2}$. Use (10.5.6) to pick $N_2 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ n \geq N_2 \implies |s_n - L_2| < \epsilon_2.$$

   Let $N = \max\{N_1, N_2\}$. We have

$$|L_1 - L_2| \leq |s_N - L_1| + |s_N - L_2| < \epsilon_1 + \epsilon_2 = \epsilon.$$

   This question shows that the limit of a sequence is unique; a sequence of real numbers cannot converge to two different real numbers.

5. Suppose that $t \in \mathbb{R} \setminus \{0\}$ and that $(t_n)_{n=1}^{\infty}$ is a sequence converging to t.

   (a) Let $\epsilon = |t|$. Since $t \neq 0$, $\epsilon > 0$. Since (t_n) converges to t, we can pick $N \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ n \geq N \implies |t_n - t| < \epsilon. \tag{10.5.7}$$

   Let $n \in \mathbb{N}$ and suppose that $n \geq N$.
   Since $n \geq N$ and (10.5.7) is true, we have $|t_n - t| < \epsilon$, i.e. $|t_n - t| < |t|$.
   If we had $t_n = 0$, then we'd obtain $|0 - t| < |t|$, a contradiction, so we must have $t_n \neq 0$.

11 Final

1. (a) **Definition.** Suppose X and Y are sets and $f : X \longrightarrow Y$ is a function.
   - We say f is *injective* **iff**

   $$\forall x_1 \in X,\ \forall x_2 \in X,\ f(x_1) = f(x_2) \implies x_1 = x_2.$$

   - We say f is *surjective* **iff**

   $$\forall y \in Y,\ \exists x \in X:\ f(x) = y.$$

   (b) Suppose X, Y, Z are sets and that $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are functions. Prove that if f and g are surjective, then $g \circ f$ is surjective.
   Solution. This was a homework problem, so you can find the solution elsewhere.

   (c) Answer your favorite THREE of the following four.
   (i) Define a function $f : \mathbb{R} \longrightarrow \mathbb{N}$ such that f is surjective.
   (ii) Define a function $g : \mathbb{Q} \longrightarrow \mathcal{P}(\mathbb{Q})$ such that g is injective.
   (iii) Define a function $h : \mathcal{P}(\mathbb{N}) \longrightarrow \mathbb{N}$ such that h is surjective.
   (iv) Define a function $j : \mathbb{Z} \longrightarrow \mathbb{N}$ such that j is injective, but not surjective.

   Solution.
   (i) $f(x) = \lceil |x| \rceil + 1$.
   (ii) $g(q) = \{q\}$.
   (iii) $h(\emptyset) = 1$; and for $A \neq \emptyset$, $h(A) = \min A$.
   (iv) $j(n) = 2n + 2$ if $n \geq 0$; $j(n) = -2n + 1$ if $n < 0$.

2. Recall the following definition.

   Definition. Suppose $(s_n)_{n=1}^{\infty}$ is a sequence.

   We say that (s_n) *converges* **iff** the following sentence is true.

$$\exists L \in \mathbb{R} : \ \forall \epsilon > 0, \ \exists N \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N \implies |s_n - L| < \epsilon.$$

   (a) Prove that the following sentence is true.

$$\exists N_0 \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq N_0 \implies 2n^2 - 888 \geq n^2.$$

   Solution. Let $N_0 = 888$. Let $n \in \mathbb{N}$. Suppose that $n \geq N_0$. Then

$$2n^2 - 888 = n^2 + (n^2 - 888) \geq n^2 + (N_0^2 - 888) = n^2 + (888^2 - 888) \geq n^2.$$

   (b) Prove that $\left(\frac{30n^3}{2n^4 - 888n^2 + n + 1001}\right)_{n=1}^{\infty}$ converges.

   Solution. Let $L = 0$. Let $\epsilon > 0$. Let $N = \max\{888, \lceil \frac{30}{\epsilon} \rceil + 1\}$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. We want to show that $\left|\frac{30n^3}{2n^4 - 888n^2 + n + 1001} - L\right| < \epsilon$, i.e. that

$$\left|\frac{30n^3}{2n^4 - 888n^2 + n + 1001}\right| < \epsilon.$$

   Since $n \geq N \geq 888$, we have $2n^2 - 888 \geq n^2$. So $2n^4 - 888n^2 \geq n^4$, and

$$2n^4 - 888n^2 + n + 1001 \geq n^4 > 0.$$

   Thus,

$$\left|\frac{30n^3}{2n^4 - 888n^2 + n + 1001}\right| = \frac{30n^3}{2n^4 - 888n^2 + n + 1001} \leq \frac{30n^3}{n^4} = \frac{30}{n}.$$

   Since $n \geq N \geq \lceil \frac{30}{\epsilon} \rceil + 1 > \frac{30}{\epsilon}$, we have $\frac{30}{n} < \epsilon$.

3. Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of real numbers, and $L' \in \mathbb{R}$.

Suppose that $(a_n)_{n=1}^{\infty}$ converges to L', and that the following sentence is true:

$$\exists K \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq K \implies b_n = 8a_n.$$

Prove that $(b_n)_{n=1}^{\infty}$ converges.

You should NOT require any of the theorems from section 10 of my notes.

Only use the definitions that were given in section 8 of my notes.

Solution.

Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences, and $L' \in \mathbb{R}$.

Suppose that $(a_n)_{n=1}^{\infty}$ converges to L', and that the following sentence is true:

$$\exists K \in \mathbb{N} : \ \forall n \in \mathbb{N}, \ n \geq K \implies b_n = 8a_n. \tag{11.1}$$

We verify that $(b_n)_{n=1}^{\infty}$ converges.

(a) Let $L = 8L'$.

(b) Let $\epsilon > 0$.

(c) Using (11.1), pick a $K \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \ n \geq K \implies b_n = 8a_n. \tag{11.2}$$

Using that $\frac{\epsilon}{8} > 0$ and $(a_n)_{n=1}^{\infty}$ converges to L', pick an $N' \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \ n \geq N' \implies |a_n - L'| < \frac{\epsilon}{8}. \tag{11.3}$$

Let $N = \max\{K, N'\}$.

(d) Let $n \in \mathbb{N}$.

(e) Suppose that $n \geq N$. We want to show that $|b_n - L| < \epsilon$, i.e $|b_n - 8L'| < \epsilon$.
Since $n \geq N \geq K$, (11.2) gives $b_n = 8a_n$. Thus,

$$|b_n - 8L'| = 8|a_n - L'|.$$

Since $n \geq N'$, (11.3) gives $|a_n - L'| < \frac{\epsilon}{8}$, so

$$|b_n - 8L'| = 8|a_n - L'| < \epsilon.$$

4. For each of the following sentences $\mathscr{S}$,
   - write down its negation NOT($\mathscr{S}$) and
   - verify either $\mathscr{S}$ or NOT($\mathscr{S}$).

(a) $\forall P : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$,
$$\Big(\exists x' \in \mathbb{R} : \ \forall y' \in \mathbb{R}, \ P(x', y') = 0\Big) \implies \Big(\forall y \in \mathbb{R}, \ \exists x \in \mathbb{R} : \ P(x, y) = 0\Big).$$

(b) $\forall P : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$,
$$\Big(\forall y' \in \mathbb{R}, \ \exists x' \in \mathbb{R} : \ P(x', y') = 0\Big) \implies \Big(\exists x \in \mathbb{R} : \ \forall y \in \mathbb{R}, \ P(x, y) = 0\Big).$$

Solution.

(a) The negation is
$$\exists P : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} :$$
$$\Big(\exists x' \in \mathbb{R} : \ \forall y' \in \mathbb{R}, \ P(x', y') = 0\Big) \text{ and } \Big(\exists y \in \mathbb{R} : \ \forall x \in \mathbb{R}, \ P(x, y) \neq 0\Big).$$

The original sentence is true. Here's the proof:
Let $P : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Suppose
$$\Big(\exists x' \in \mathbb{R} : \ \forall y' \in \mathbb{R}, \ P(x', y') = 0\Big). \tag{11.4}$$

We want to show $\Big(\forall y \in \mathbb{R}, \ \exists x \in \mathbb{R} : \ P(x, y) = 0\Big)$.

i. Let $y \in \mathbb{R}$.
ii. Using (11.4), pick $x' \in X$ such that
$$\forall y' \in \mathbb{R}, \ P(x', y') = 0. \tag{11.5}$$
Let $x = x'$.
iii. Since $x = x'$, we see $P(x, y) = P(x', y)$. (11.5) tells us that $P(x', y) = 0$.

(b) The negation is true and given by
$$\exists P : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} :$$
$$\Big(\forall y' \in \mathbb{R}, \ \exists x' \in \mathbb{R} : \ P(x', y') = 0\Big) \text{ and } \Big(\forall x \in \mathbb{R} : \ \exists y \in \mathbb{R} : \ P(x, y) \neq 0\Big).$$

Here's the proof...
Define $P : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $P(x, y) = x - y$. We must verify
$$\Big(\forall y' \in \mathbb{R}, \ \exists x' \in \mathbb{R} : \ P(x', y') = 0\Big) \text{ and } \Big(\forall x \in \mathbb{R} : \ \exists y \in \mathbb{R} : \ P(x, y) \neq 0\Big).$$

For the first clause: let $y' \in \mathbb{R}$; let $x' = y'$; then $P(x', y') = x' - y' = y' - y' = 0$.
For the second clause: let $x \in \mathbb{R}$; let $y = x-1$; then $P(x, y) = x-y = x-(x-1) = 1 \neq 0$.

5. Recall the following **Definitions / Theorem.**

- Fix $n \in \mathbb{N}$.
- In class, we defined a relation on $\mathbb{Z}$ by declaring $a \sim_n b$ **iff**

$$\exists d \in \mathbb{Z}: \; a - b = dn.$$

- On the homework, you proved that $\sim_n$ is an equivalence relation.
- Given $a \in \mathbb{Z}$, we write $[a]_n$ for the equivalence class

$$\Big\{ b \in \mathbb{Z}: \; a \sim_n b \Big\},$$

- and we write $\mathbb{Z}/n$ for the set of equivalence classes

$$\Big\{ [a]_n : \; a \in \mathbb{Z} \Big\}.$$

Let j be the function defined by $j : \mathbb{Z} \longrightarrow \mathbb{Z}/3 \times \mathbb{Z}/4$, $a \longmapsto ([a]_3, [a]_4)$.

Let $\approx$ be the relation on $\mathbb{Z}$ defined by declaring $m_1 \approx m_2$ **iff** $j(m_1) = j(m_2)$.

It is true that $\approx$ is an equivalence relation; you do not need to prove this.

Use the notation $[a]_\approx$ for the equivalence class $\{b \in \mathbb{Z}: \; a \approx b\}$.

(a) Fix $n \in \mathbb{N}$ and prove that $\sim_n$ is transitive, i.e. redo some of your homework.

(b) Describe explictly the set of equivalence classes $\mathbb{Z}/\approx$, i.e. write down the set of equivalence classes in such a way that I know you completely understand the equivalence relation $\approx$.

Solution.

(a) This was a homework problem, so you can find the solution elsewhere.

(b) $\{[0]_\approx, [1]_\approx, [2]_\approx, [3]_\approx, [4]_\approx, [5]_\approx, [6]_\approx, [7]_\approx, [8]_\approx, [9]_\approx, [10]_\approx, [11]_\approx\}$.

6. Let $\equiv$ be the relation on $\mathbb{R} \times \mathbb{R}$ defined by declaring $(x_1, x_2) \equiv (y_1, y_2)$ **iff**

$$\exists \lambda \in \mathbb{R} \setminus \{0\} : \ (\lambda x_1, \lambda x_2) = (y_1, y_2).$$

It is true that $\equiv$ is an equivalence relation; you do not need to prove this.

Fix $a, b, c, d \in \mathbb{R}$, and try to define $f : (\mathbb{R} \times \mathbb{R})/{\equiv} \longrightarrow (\mathbb{R} \times \mathbb{R})/{\equiv}$ by

$$f([(x_1, x_2)]) = [(ax_1 + bx_2, cx_1 + dx_2)].$$

Prove that f is well-defined.

Solution.

Suppose $[(x_1, x_2)] = [(y_1, y_2)]$. Then $(x_1, x_2) \equiv (y_1, y_2)$.

Pick a $\lambda \in \mathbb{R} \setminus \{0\}$ such that $(\lambda x_1, \lambda x_2) = (y_1, y_2)$. We have

$$(\lambda(ax_1 + bx_2), \lambda(cx_1 + dx_2)) = (a(\lambda x_1) + b(\lambda x_2), c(\lambda x_1) + d(\lambda x_2)) = (ay_1 + by_2, cy_1 + dy_2).$$

Thus, $(ax_1 + bx_2, cx_1 + dx_2) \equiv (ay_1 + by_2, cy_1 + dy_2)$, which gives

$$[(ax_1 + bx_2, cx_1 + dx_2)] = [(ay_1 + by_2, cy_1 + dy_2)],$$

and so f is well-defined.