# Math 32B
# Calculus of Several Variables

## Midterm 2

**Instructions:** You have 50 minutes to complete this exam. There are four questions, worth a total of 45 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ______________________________

Student ID number: ______________________

Discussion: ___________________________

| Question | Points | Score |
|---|---|---|
| 1 | 13 | |
| 2 | 9 | |
| 3 | 12 | |
| 4 | 11 | |
| Total: | 45 | |

**Problem 1.**

(a) [2pts.] Let $G(u, v) = (u + v, v^2)$. Let $c$ be a fixed constant.
What is the image under $G$ in the $(x, y)$-plane of the vertical line $u = c$?

> **Solution:** For all $v$, $G(c, v) = (c+v, v^2)$ lies on the curve the curve $y = (x-c)^2$.

(b) [2pts.] Again, let $G(u, v) = (u + v, v^2)$. Let $d$ be a fixed constant.
What is the image under $G$ in the $(x, y)$-plane of the horizontal line $v = d$?

> **Solution:** For all $u$, $G(u, d) = (u + d, d^2)$ lies on the line $y = d^2$.

(c) [2pts.] Now let $G(u, v) = (e^{uv^2}, u - v)$. Calculate the Jacobian of $G$.

> **Solution:** $J(G) = \begin{pmatrix} v^2 e^{uv^2} & 2uve^{uv^2} \\ 1 & -1 \end{pmatrix}$, $\det J(G) = -(2u + v)ve^{uv^2}$.

(d) [7pts.] Let $\mathcal{D}$ be the region in the first quadrant $x, y \geq 0$ enclosed by

$$y = x, \ y = 4x, \ y = \frac{1}{x}, \ y = \frac{9}{x}.$$

Find the area of $\mathcal{D}$.

> **Solution:** Let $G(u, v) = (\frac{v}{u}, uv)$. Then $\mathcal{D}$ is the image of $[1, 2] \times [1, 3]$.
> The Jacobian of $G$ is $\begin{pmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ v & u \end{pmatrix}$, so the absolute value of its determinant is $\frac{2v}{u}$.
> The area of $\mathcal{D}$ is $\int_{u=1}^{2} \int_{v=1}^{3} \frac{2v}{u} \, dv \, du = 8 \ln 2$.

**Problem 2.**

(a) [4pts.] Let $C$ be the oriented curve parametrized by $\mathbf{r}(t) = (t^2, t, 2t)$, $0 \leq t \leq \pi$.
Calculate $\int_C 2xy \, dx + (x^2 + e^y \sin z) \, dy + e^y \cos z \, dz$.

> **Solution:** $f(x, y, z) = x^2y + e^y \sin z$ is a potential for $(2xy, x^2 + e^y \sin z, e^y \cos z)$.
> The integral is equal to $f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(\pi^2, \pi, 2\pi) - f(0, 0, 0) = \pi^5$.

(b) [5pts.] Let $C$ be the oriented curve which lies in the first quadrant of the $xy$-plane, is described by $x^2 + 4y^2 = 4$ and $z = 0$, starts at $(2, 0, 0)$, and ends at $(0, 1, 0)$.
Let $\mathbf{F}(x, y, z) = (0, 3xy, 0)$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

> **Solution:** Let $\mathbf{r}(t) = (2 \cos t, \sin t, 0)$. Then $\mathbf{r}'(t) = (-2 \sin t, \cos t, 0)$.
> So $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 6 \cos^2 t \sin t$, and $\int_C \mathbf{F} \cdot d\mathbf{r} = [-2 \cos^3 t]_0^{\pi/2} = 2$.

**Problem 3.**

All of the vector fields in this question are defined on $\mathbb{R}^3 - (z\text{-axis})$.

In each of the following parts, say whether $\mathbf{F}(x, y, z)$ is conservative on $\mathbb{R}^3 - (z\text{-axis})$.

(a) [3pts.] $\mathbf{F}(x, y, z) = (z, x, y)$.

> **Solution:** $\nabla \times \mathbf{F} = (1, 1, 1) \neq \mathbf{0}$ so $\mathbf{F}$ is not conservative on $\mathbb{R}^3 - (z\text{-axis})$.

(b) [3pts.] $\mathbf{F}(x, y, z) = (x, y^2, z^3)$.

> **Solution:** $f(x, y, z) = \frac{x^2}{2} + \frac{y^3}{3} + \frac{z^4}{4}$ is a potential for $\mathbf{F}$,
> so $\mathbf{F}$ is conservative on $\mathbb{R}^3 - (z\text{-axis})$.

(c) [6pts.] $\mathbf{F}(x, y, z) = (x + yz, y^2 + xz, z^3 + xy) + \frac{(-y,x,0)}{x^2+y^2}$.

> **Solution:** Let $f(x, y, z) = xyz + \frac{x^2}{2} + \frac{y^3}{3} + \frac{z^4}{4}$
> and let $\mathbf{V}$ be the vortex vector field. Then $\mathbf{F} = \nabla f + \mathbf{V}$.
> Let $C$ be the unit circle in the $xy$-plane oriented counter-clockwise.
> Then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} + \int_C \mathbf{V} \cdot d\mathbf{r} = 0 + 2\pi = 2\pi \neq 0$.
> Since $\mathbf{F}$ has a non-zero circulation, $\mathbf{F}$ is not conservative on $\mathbb{R}^3 - (z\text{-axis})$.

**Problem 4.**

(a) [6pts.] Calculate

$$\iint_T (x + y + z)\, dS$$

where $T$ is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

> **Solution:** The vertices of the triangle lie on the plane is $x + y + z = 1$.
> Let $G(x, y) = (x, y, 1 - x - y)$, for $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$.
> $G_x \times G_y = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1)$. $\int_{x=0}^{1} \int_{y=0}^{1-x} \|(1, 1, 1)\|\, dy\, dx = \frac{\sqrt{3}}{2}$.

(b) [5pts.] Suppose that $f(z)$ is a continuously differentiable positive function.
By rotating the graph $x = f(z)$ around the $z$-axis we obtain a surface

$$x^2 + y^2 = f(z)^2.$$

Find a continuously varying non-zero normal $\mathbf{N}(x, y, z)$ in terms of $f(z)$ and $f'(z)$.

> **Solution:** Parametrize the surface by $G(\theta, z) = (f(z)\cos\theta, f(z)\sin\theta, z)$.
> Then $G_\theta \times G_z = (f(z)\cos\theta, f(z)\sin\theta, -f(z)f'(z))$.
> So $\mathbf{N}(x, y, z) = (x, y, -f(z)f'(z))$ is normal to the surface at $(x, y, z)$.