LAST TIME : CONSERVATIVE VECTOR FIELDS.

- $\underline{F}$ CONSERVATIVE ON D

  $\overset{\text{DEFN}}{\equiv}$ CAN FIND A FUNCTION f ON D WITH $\nabla f = \underline{F}$.

- $\underline{F}$ CONSERVATIVE ON D $\Rightarrow$ $\nabla \times \underline{F} = \underline{0}$.

  $\nabla \times \underline{F} \neq 0$ $\Rightarrow$ $\underline{F}$ NOT CONSERVATIVE ON D.

- $\underline{F}$ CONSERVATIVE ON D $\Leftrightarrow$ ALL CIRCULATIONS OF $\underline{F}$ ARE 0.

  $\underline{F}$ HAS A NON-ZERO CIRCULATION $\Leftrightarrow$ $\underline{F}$ IS NOT CONSERVATIVE ON D.

- SUPPOSE $\nabla \times \underline{F} = \underline{0}$.

  NON-ZERO CIRCULATIONS REQUIRE WRAPPING AROUND A HOLE IN D.

  IF D HAS NO HOLES, THEN $\underline{F}$ IS CONSERVATIVE ON D.

- WHEN $\underline{F}$ IS CONSERVATIVE ON D, FINDING A POTENTIAL IS OFTEN POSSIBLE BY "INTEGRATING" WITH RESPECT TO EACH VARIABLE, ie SOLVING

$$\frac{\partial f}{\partial x} = F_1$$
$$\frac{\partial f}{\partial y} = F_2$$
$$\frac{\partial f}{\partial z} = F_3.$$

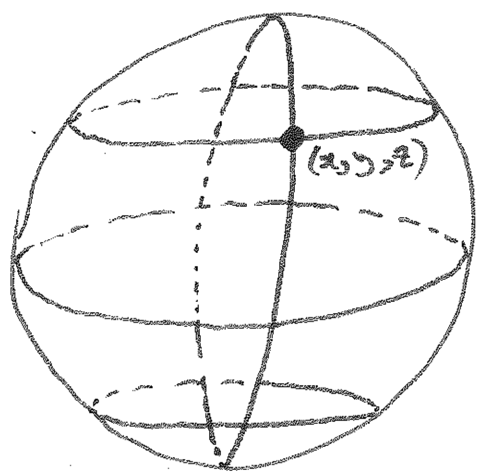
SURFACES.

WHAT IS A SURFACE? A 2-D SHAPE.

EXAMPLE: THE SURFACE OF THE EARTH.

WHAT MAKES IT 2D?

IF YOU LOOK AT A MAP OF THE WORLD, IT'LL USE 2 COORDINATES: LATITUDE AND LONGITUDE.



LATITUDE : NORTH / SOUTH

LONGITUDE : EAST / WEST.

SUPPOSE EARTH HAS RADIUS R.

LET $G(\phi, \theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$

A LINE OF LATITUDE IS IMAGE OF ϕ = CONSTANT.
A LINE OF LONGITUDE IS IMAGE OF θ = CONSTANT, $0 \leq \phi \leq \pi$.

THE SPHERE IS PARTICULARLY SYMMETRIC SO YOU CAN WRITE DOWN

a) A TANGENT VECTOR AT (x,y,z) POINTING ALONG THE LINE OF LONGITUDE THROUGH (x,y,z):

$$(xz, yz, -x^2-y^2).$$

b) A TANGENT VECTOR AT (x,y,z) POINTING ALONG THE LINE OF LATITUDE THROUGH (x,y,z):

$$(-y, x, 0).$$

c) A NORMAL VECTOR AT (x,y,z):

$$(x,y,z).$$

CAN ALSO USE PARAMETRIZATION:

$$G_\phi(\phi,\theta), \quad G_\theta(\phi,\theta), \quad G_\phi(\phi,\theta) \times G_\theta(\phi,\theta).$$

$$\begin{pmatrix} R\cos\phi\cos\theta \\ R\cos\phi\sin\theta \\ -R\sin\phi \end{pmatrix} \times \begin{pmatrix} -R\sin\phi\sin\theta \\ R\sin\phi\cos\theta \\ 0 \end{pmatrix} = R^2\sin\phi \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}.$$

LAST TIME : PARAMETRIZED SURFACE

EXAMPLES : A SPHERE OF RADIUS R : $x^2+y^2+z^2=R^2$

$$G(\phi,\theta) = (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi)$$

CYLINDER OF RADIUS R : $x^2+y^2=R^2$

$$G(\theta,z) = (R\cos\theta,\ R\sin\theta,\ z)$$

A GRAPH $z=f(x,y)$

$$G(x,y) = (x,y,f(x,y))$$

HELICOID

$$G(u,v) = (u\cos v,\ u\sin v,\ v).$$

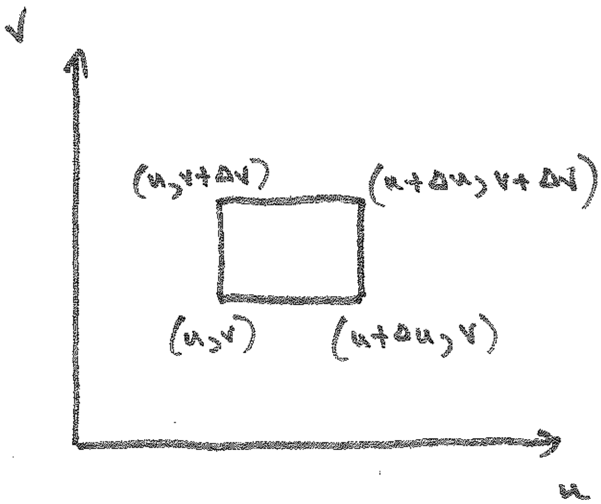
GIVEN A PARAMETRIZED SURFACE $G(u,v)$

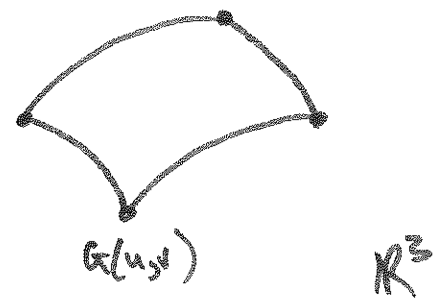
THE TEXTBOOK WRITES

$\underline{T}_u$ FOR $G_u(u,v)$ TANGENT VECTOR POINTING ALONG $v=$ CONST.

$\underline{T}_v$ FOR $G_v(u,v)$ .... $u=$ CONST.

$\underline{N}$ FOR $\underline{T}_u \times \underline{T}_v$ NORMAL VECTOR. NOT NECESSARILY A UNIT VECTOR.





$$\text{AREA} \approx \left\| \Big(G(u+\Delta u, v) - G(u,v) \Big) \times \Big(G(u, v+\Delta v) - G(u,v) \Big) \right\|$$

$$\approx \left\| G_u(u,v)\,\Delta u \times G_v(u,v)\,\Delta v \right\|$$

$$= \| \underline{N} \| \, \Delta u \, \Delta v .$$

$$\iint_S f(x,y,z)\, dS = \iint_D f\big(G(u,v)\big) \, \|\underline{N}(u,v)\| \, du\, dv .$$

WHERE $G(u,v)$ IS A PARAMETRIZATION OF S WITH PARAMETER DOMAIN D.

E.G. AREA OF A SPHERE OF RADIUS R $= \displaystyle\int_0^{2\pi} \int_0^{\pi} R^2 \sin\phi \, d\phi \, d\theta$

$$= 2\pi \Big[R^2 (-\cos\phi) \Big]_0^{\pi} = 4\pi R^2 .$$

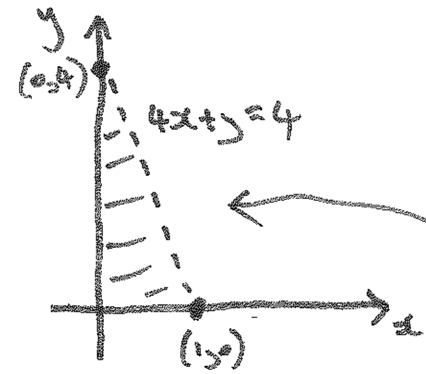
EXAMPLE: CALCULATE

$$\iint_S 3 - z \, dS$$

WHERE S IS THE TRIANGLE WITH VERTICES

$(0,0,3)$, $(1,0,2)$, $(0,4,1)$

METHOD 1:



TRIANGLE IS GRAPH OVER

EQU. OF TRIANGLE / PLANE?

$$\left(\begin{pmatrix}0\\0\\3\end{pmatrix} - \begin{pmatrix}0\\4\\1\end{pmatrix}\right) \times \left(\begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}0\\4\\1\end{pmatrix}\right) = 2\begin{pmatrix}2\\1\\2\end{pmatrix}$$

EQU. $2x + y + 2z = 6$

$$G(x,y) = \left(x, y, \frac{6-2x-y}{2}\right)$$

$$\| G_x \times G_y \| = \frac{3}{2}$$

$$\int_0^1 \int_0^{4-4x} \frac{2x+y}{2} \cdot \frac{3}{2} \, dy \, dx = 3$$

METHOD 2 :

$$G(u,v) = (1-u)(0,0,3) + u\left[(1-v)(1,0,2) + v(0,4,1)\right]$$

$$G_u(u,v) = -(0,0,3) + (1-v)(1,0,2) + v(0,4,1)$$
$$= (1-v, 4v, -1-v)$$

$$G_v(u,v) = u\left[-(1,0,2) + (0,4,1)\right] = (-u, 4u, -u)$$

$$G_u \times G_v = 2u\begin{pmatrix}2\\1\\2\end{pmatrix}. \qquad \|G_u \times G_v\| = 6u$$

$$\int_0^1\int_0^1 \Big(3 - \big(3(1-u) + 2u(1-v) + uv\big)\Big)\, 6u \; du\, dv$$

$$= \int_0^1\int_0^1 6u^2(1+v)\; du\, dv = 3.$$

[redacted]

METHOD 3:

$$G(u,v) = (0,0,3) + u\big((1,0,2)-(0,0,3)\big) + v\big((0,4,1)-(0,0,3)\big)$$

$$G_u \times G_v = \big((1,0,2)-(0,0,3)\big) \times \big((0,4,1)-(0,0,3)\big) = 2(2,1,2)$$

$$\|G_u \times G_v\| = 6$$

$$\int_0^1 \int_0^{1-u} \big(3-(3-u-2v)\big)\, 6 \; dv\, du$$

$$= \int_0^1 \int_0^{1-u} 6(u+2v)\; dv\, du = 3.$$

A SURFACE IS ORIENTED IF A CONTINUOUSLY VARYING UNIT NORMAL VECTOR $\underline{n}(P)$ IS SPECIFIED AT EACH POINT IN S.

NOT ALL SURFACES ARE ORIENTABLE ...

$$G(u,v) = \left(\cos u\left(1+v\cos\frac{u}{2}\right),\ \sin u\left(1+v\cos\frac{u}{2}\right),\ v\sin\frac{u}{2}\right)$$

$$0 \le u \le 2\pi,\quad -\frac{1}{2} \le v \le \frac{1}{2}.$$

NOTICE $\quad G(0,v) = G(2\pi,-v)$

IN PARTICULAR, $\quad G(0,0) = G(2\pi,0) = (1,0,0).$

$$G_u(u,v) = \Big(-\sin u\left(1+v\cos\frac{u}{2}\right) - \frac{v}{2}\cos u\sin\frac{u}{2}$$
$$\cos u\left(1+v\cos\frac{u}{2}\right) - \frac{v}{2}\sin u\sin\frac{u}{2},\ \frac{v}{2}\cos\frac{u}{2}\Big)$$

$$G_v(u,v) = \left(\cos u\cos\frac{u}{2},\ \sin u\cos\frac{u}{2},\ \sin\frac{u}{2}\right)$$

SO $\quad G_u(0,0) = (0,1,0) = G_u(2\pi,0)$

AND $\quad G_v(0,0) = (1,0,0) = -(-1,0,0) = -G_v(2\pi,0)$

THUS, $\quad G_u(0,0) \times G_v(0,0) = -\,G_u(2\pi,0) \times G_v(2\pi,0).$

THE MOBIUS BAND IS NOT ORIENTABLE!

$$\iint_S \underline{F} \cdot d\underline{S} \; := \; \iint_S \underline{F} \cdot \underline{n} \, dS$$

WHERE $\underline{n}$ IS UNIT NORMAL TO S.

$$\iint_D \underline{F}(G(u,v)) \cdot \underline{n}(G(u,v)) \; \|\underline{N}(u,v)\| \, du \, dv$$

$$= \iint_D \underline{F}(G(u,v)) \cdot \underline{N}(u,v) \, du \, dv.$$

④

$$\iint_S \underline{F} \cdot \underline{dS}$$

W/ OUTWARD POINTING NORMAL

WHERE S IS THE SPHERE OF RADIUS R,

$$\underline{F}(x,y,z) = \frac{(x,y,z)}{\rho^3}.$$

$$G(\phi,\theta) = (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi)$$

$$\underline{N} = G_\phi \times G_\theta = R^2\sin\phi \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

$$\underline{F}(G(\phi,\theta)) \cdot \underline{N} = \frac{\begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix}}{R^3} \cdot \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} R^2\sin\phi$$

$$= \sin\phi.$$

$$\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta = 2\pi\Big[-\cos\phi\Big]_0^{\pi} = 4\pi.$$

(5)

$$\iint_S \underline{F} \cdot d\underline{S}$$

WHERE S IS CYLINDER OF RADIUS R, $0 \leq z \leq 1$,

OUTWARD POINTING NORMAL,

$$\underline{F}(x,y,z) = \frac{(x,y,0)}{r^2}.$$

$$G(\theta,z) = (R\cos\theta, R\sin\theta, z)$$

$$G_\theta \times G_z = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \end{pmatrix}$$

$$\underline{F}(G(\theta,z)) \times \underline{N} = \frac{\begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \end{pmatrix}}{R^2} \cdot \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \end{pmatrix} = 1$$

$$\int_0^1 \int_0^{2\pi} 1 \, d\theta \, dz = 2\pi.$$